# Poisson Random Measures and Multitype Branching Processes

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# 1. Definition of the model - Immigration

- Let $0 < T_1 < T_2 < \cdots$ denote random time-points generated by a Poisson random measure (PMR) $\Pi(0,t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \le t\}}, t \ge 0$, with local intensity $r(t) > 0$ and mean measure $R(t) = \int_0^t r(x)dx$. Then, $\mathbf{P}\{\Pi(0,t) = k\} = e^{-R(t)} R^k(t)/k!$ for $k = 0, 1....$
- Let $\mathbf{I}_k = (I_{k1}, \ldots, I_{kd}), k = 1, 2 \cdots$ , be i.i.d. non-negative integer-valued random vectors with a multidimensional p.g.f. $g(\mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{I}_k}\} = \sum_{\boldsymbol{\alpha}} \mathbf{P}\{\mathbf{I}_k = \boldsymbol{\alpha}\}\mathbf{s}^{\boldsymbol{\alpha}}$, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, $\mathbf{s} = (s_1, ..., s_d)$, $|\ \mathbf{s}\ | \le \mathbf{1}$, $\mathbf{s}^{\boldsymbol{\alpha}} = \sqcap_{i=1}^{d} s_i^{\alpha_i}$.
- We consider the marked point process $\{(T_k, \mathbf{I}_k), k = 1, 2...\}$. The vector $\mathbf{I}_k$ is interpreted as the number of immigrants that join the population at time $T_k$.

# 2. Definition of the model - Branching

- Define the multitype branching process $\mathbf{Z} = \{\mathbf{Z}_i(t) = (Z_{i1}(t), Z_{i2}(t), \ldots, Z_{id}(t)), i = 1, ..., d; t \geq 0\}$, where $Z_{ij}(t)$ denotes the number of type-$j$ individuals (cells, particles) at time $t$ produced by a single type-$i$ individual born at $t = 0$, $i, j = 1, \ldots, d$, and **assume that individuals evolve independently of each other.**
- Let $F_i(t; \mathbf{s}) = \sum_{\alpha \in \mathbf{N}^d} \mathbf{P}\{\mathbf{Z}_i(t) = \alpha\}\mathbf{s}^{\alpha}$, with $F_i(0, \mathbf{s}) = s_i$, be the corresponding multitype p.g.f. Define the vector $\mathbf{F}(t; \mathbf{s}) = (F_1(t; \mathbf{s}), F_2(t; \mathbf{s}), \ldots, F_d(t; \mathbf{s}))$.
- Let $\widetilde{\mathbf{Z}} = \{\widetilde{\mathbf{Z}}_k(t) = (\widetilde{Z}_{k1}(t), ..., \widetilde{Z}_{kd}(t)); t \geq 0; k = 1, 2, \ldots\}$ be i.i.d. copies of $\mathbf{Z}$, but with initial conditions $\widetilde{\mathbf{Z}}_k(0) = \mathbf{I}_k$. Therefore, $\mathbf{E}\{\mathbf{s}^{\widetilde{\mathbf{Z}}_k(t)}\} = g(\mathbf{F}(t; \mathbf{s}))$ **because of the independence of the individual evolutions.**
- We assume that the sets $\widetilde{\mathbf{Z}}$ and $\{\Pi(0, t), t \geq 0\}$ are independent.

# 3. PRM and Branching Processes

- Define the process

$$\mathbf{Y}(t) = \sum_{k=1}^{\Pi(0,t)} \widetilde{\mathbf{Z}}_k(t - T_k)\mathbf{1}_{\{\Pi(t)>0\}},\ t \geq 0,\ \mathbf{Y}(0) = \mathbf{0}.$$

- Its first increment occurs when the first batch of $\mathbf{I}_1$ immigrants enters the population at time $T_1$, each of which evolves in accordance with a process $\mathbf{Z}$.

- A second batch of $\mathbf{I}_2$ immigrants arrives at time $T_2$, etc.

- We refer to $\mathbf{Y} = \{\mathbf{Y}(t) = (Y_1(t), ..., Y_d(t)), t \geq 0\}$ as a *Multitype Branching Process with Non-Homogeneous Poisson Immigration (MBPwNHPI)*.

**Theorem A.** The p.g.f.

$$\Phi(t;\mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{Y}(t)} | \mathbf{Y}(t) = 0\}$$

satisfies the equation:

$$\Phi(t;\mathbf{s}) = \exp\left\{-\int_0^t r(t-x)[1 - g(\mathbf{F}(x;\mathbf{s}))]dx\right\}, \Phi(0;\mathbf{s}) = \mathbf{1}.$$

**Remark.** This formula is valid for a broad class of branching processes in which **individuals evolve independently of each other.** Such processes include multitype Markov, Bellman-Harris, Sevastyanov or Crump-Mode-Jagers branching models, which are described in various monographs: Harris (1963), Sevastyanov 1971), Mode (1972), Athreya and Ney (1972), Jagers (1975), and Asmussen and Hering (1983).

# 5. PRM and Multitype Markov Branching Processes

- We investigate the asymptotic behaviour of $\mathbf{Y}(t)$ when $\mathbf{Z}(t)$ is a critical multitype Markov branching process and when the intensity $r(t)$ is a regularly varying function (r.v.f.), i.e.:

  $r(t) = L(t)t^{\theta}$, where $\theta \in \mathbb{R}$ and $L(t)$ is a $s.v.f.$ as $t \to \infty$.

- Depending on the asymptotic rate of $r(t)$ we study
  - asymptotic behaviour of first and second moments of $\mathbf{Y}(t)$;
  - convergence of its probabilities of non-extinction;
  - limiting distributions.

# 6. History

- The first branching process model with immigration was introduced and investigated by Sevastyanov (1957) in the single-type continuous-time Markov case and when the times of immigration form an homogeneous Poisson process.
- Jagers (1968) generalized this model to Bellman-Harris branching processes. The same setting was subsequently investigated by Pakes (1972), Radcliffe (1972), Pakes and Kaplan (1974).
- Sevastyanov branching processes with homogeneous Poisson immigration were considered by Yanev (1972).
- Multitype Markov branching processes with homogeneous Poisson immigration were considered by Polin (1977) and Sagitov (1982), among others.

# 7. Motivation - by Applications

Recently, single-type age-dependent branching processes with immigration occurring according to an inhomogeneous Poisson process have been considered to describe the evolution of cell populations:

- *Yakovlev A.Y., Yanev N.M.* Branching stochastic processes with immigration in analysis of renewing cell populations// Math. Biosci. 2006. V. 203. P. 37-63.
- *Yanev N.M.* Branching Processes in Cell Proliferation Kinetics. In: M. Gonzalez et al. (Eds.), Workshop on Branching Processes and Their Applications. LN in Statistics 197, 2010, 159-179.
- *Hyrien, O., Peslak, S.A., Yanev, N.M., Palis, J.* (2015). Stochastic modeling of stress erythropoiesis using a two-type age-dependent branching process with immigration. J. Math. Biol.70:1485-1521
- *Hyrien O., Yanev N.M., Jordan C.T. (2015) A test of homogeneity for age-dependent branching processes with immigration. Electronic J. Statistics. Vol. 9, 898–925.*

# 8. Motivation - by Theory - 1

For many years, we have studied several, related classes of branching processes with migration, independent immigration and state-dependent immigration.

- *Yanev N. M. and Mitov K. V.* Critical Branching Processes with Nonhomogeneous Migration// Ann. Probab. 1985. V. 13(3). P. 923-933.
- *Mitov K.V., Vatutin V.A., Yanev N.M.* Continuous-time branching processes with decreasing state-dependent immigration// Adv. Appl. Prob. 1984. V. 16. P. 697-714.
- *Mitov K.V., Yanev N.M.* Bellman-Harris branching processes with state-dependent immigration// J. Appl. Prob. 1985. V. 22. P. 757-765.

# 9. Motivation - by Theory - 2

Single-type Sevastyanov branching processes with immigration arising from Poisson random measures were investigated in the following papers:

- *Mitov K.V., Yanev N. M. (2013) Sevastyanov branching processes with non-homogeneous Poisson immigration. Proceedings of Steklov Mathematical Institute, V. 282, 181-194*
- *Hyrien, O., Mitov K. M. , Yanev N. M. (2016). Supercritical Sevastyanov branching processes with non-homogeneous Poisson immigration. Eds. I.M. del Puerto, et al., "Branching Processes and their Applications", Lecture Notes in Statistics, vol. 219, 151–166, Springer, New York.*
- *Hyrien, O., Mitov K. M. , Yanev N. M. (2017) Subcritical Sevastyanov branching processes with non-homogeneous Poisson immigration. J. Appl. Prob. 54, 569-587*.

# 10. Multitype Markov branching processes with non-homogeneous Poisson immigration

- From now on, we assume that the lifespan $\tau_i$ and the offspring vector $\nu_i$ are independent, $G_i(t) = 1 - e^{-t/\mu_i}$, and $h_i(\mathbf{s}) = \mathbf{E}\mathbf{s}^{\boldsymbol{\nu}_i} = \sum_{\boldsymbol{\alpha}\in\mathbf{N}^d} p_{i\boldsymbol{\alpha}}\mathbf{s}^{\boldsymbol{\alpha}}$, $i = 1, 2, \ldots, d$, which implies that $\mathbf{Z}$ is a multitype Markov branching process, and

$$\frac{\partial}{\partial t}\mathbf{F}(t;\mathbf{s}) = \mathbf{f}(\mathbf{F}(t;\mathbf{s})),\ \ \frac{\partial}{\partial t}\mathbf{F}(t;\mathbf{s}) = \sum_{i=1}^{d} f_i(\mathbf{s})\frac{\partial}{\partial s_i}\mathbf{F}(t;\mathbf{s}), \mathbf{F}(0;\mathbf{s}) = \mathbf{s}$$

  where $f_i(\mathbf{s}) = [h_i(\mathbf{s}) - s_i]/\ \mu_i$ are infinitesimal g.f. and $\mathbf{f}(\mathbf{s}) = (f_1(\mathbf{s}), f_2(\mathbf{s}), \ldots, f_d(\mathbf{s}))$.

- Under these assumptions, $\mathbf{Y}(t)$ is a multitype Markov branching processes with non-homogeneous Poisson immigration (MMBPwNHPI).

# 11. Multitype Markov Branching Processes - Notation

Let $A_{ij}(t) = \mathbf{E}\{Z_{ij}(t)\} = \frac{\partial F_i(t;\mathbf{s})}{\partial s_j}\Big|_{\mathbf{s}=\mathbf{1}}, 1 \leq i,j,k \leq d,$

$$B^i_{jk}(t) = \mathbf{E}\{Z_{ij}(t)(Z_{ik}(t) - \delta_{jk})\} = \frac{\partial^2 F_i(t;\mathbf{s})}{\partial s_j \partial s_k}\Big|_{\mathbf{s}=\mathbf{1}}.$$

Introduce the matrix of first infinitesimal characteristics $\mathbf{a} = \|a_{ij}\|$ where $a_{ij} = \frac{\partial f_i(\mathbf{s})}{\partial s_j}\Big|_{\mathbf{s}=\mathbf{1}}$, and the second factorial infinitesimal characteristics $b^i_{jk} = \frac{\partial^2 f_i(\mathbf{s})}{\partial s_j \partial s_k}\Big|_{\mathbf{s}=\mathbf{1}}, \; 1 \leq i,j,k \leq d.$

Then, $\mathbf{A}(t) = \|A_{ij}(t)\| = \exp\{\mathbf{a}t\} = \sum_{n=0}^{\infty} \frac{\mathbf{a}^n t^n}{n!}$.

Assume that $\mathbf{a}$ is an irreducible matrix. Write $\rho$ for its Perron-Frobenius root. The associated right and left eigenvectors $\mathbf{u} = (u_1, \ldots, u_d)$ and $\mathbf{v} = (v_1, \ldots, v_d)$ can be chosen positive, with $u_1 > 0$ and $v_1 > 0$, and normalized such that $\sum_{i=1}^d u_i = 1$ and $\sum_{i=1}^d u_i v_i = 1$.

Define $m_i = \frac{\partial g(\mathbf{s})}{\partial s_i}\Big|_{\mathbf{s}=\mathbf{1}}, \beta_{ij} = \frac{\partial^2 g(\mathbf{s})}{\partial s_i \partial s_j}\Big|_{\mathbf{s}=\mathbf{1}}$ - the first and the second factorial moments of the immigration component.

# 12. MMBPwNHPI - Equations for moments

- Put $M_i(t) = \mathbf{E}Y_i(t)$ and $C_{ij}(t) = \mathbf{Cov}\{Y_i(t), Y_j(t)\}$, $1 \leq i, j \leq d$, where $V_k(t) = C_{kk}(t) = \mathbf{Var}\, Y_k(t)$, $k = 1, ..., d$.
- Then,

$$M_i(t) = \frac{\partial \log \Phi(t; \mathbf{s})}{\partial s_i}\Big|_{\mathbf{s}=\mathbf{1}} = \sum_{k=1}^{d} m_k \int_0^t r(t-x) A_{ki}(x)dx,$$

$$\begin{aligned} C_{ij}(t) &= \frac{\partial^2 \log \Phi(t; \mathbf{s})}{\partial s_i \partial s_j}\Big|_{\mathbf{s}=\mathbf{1}} = b \sum_{k=1}^{d} m_k \int_0^t r(t-x) B_{ij}^k(x)dx \\ &+ \sum_{k=1}^{d} \sum_{l=1}^{d} \beta_{kl} \int_0^t r(t-x) A_{ki}(x) A_{lj}(x)dx, \end{aligned}$$

where $b = \sum_{i,j,k} v_i b_{jk}^i u_j u_k$.

# 13. MMBPwNHPI - Basic assumptions

- The local characteristics of reproduction $\{a_{ij},\ b^i_{jk}\}$ and of immigration $\{m_i, \beta_{ij}\}$ are finite
- The Markov process **Z** is irreducible and critical, i.e. the Perron-Frobenius root $\rho = 0$.
- We investigate the asymptotic behaviour of $\mathbf{Y}(t)$ when the local intensity of the PRM, $r(t) = L(t)t^{\theta}$, $\theta \in \mathbb{R}$, is a r.v.f. bounded on the finite intervals and $L(t)$ is a s.v.f. as $t \to \infty$.

# 14. MMBPwNHPI - Means, variances, covariances and correlations

**Theorem (1)**

*Under the Basic assumptions, and as $t \to \infty$:*
$M_i(t) \sim Cv_i R(t), \; C = \sum_{k=1}^{d} m_k u_k, \; 1 \le i \le d,$
$C_{ij}(t) \sim bCv_i v_j \int_0^t R(x)dx, \;\; 1 \le i, j \le d, \, t \to \infty.$

**Corollary 1.** If $\rho_{ij}(t) = \mathbf{Cor}\{Y_i(t), Y_j(t)\}, i \ne j, 1 \le i, j \le d$, are the correlation coefficients then $\lim_{t\to\infty} \rho_{ij}(t) = 1.$

**Remark 1:**

- If $\theta = 0$ then $R(t)$ is a s.v.f. If additionally $R(t) \to R < \infty$, then $C_{ij}(t) \sim bCv_i v_j Rt$ (similar to the process without immigration).
- If $R(t) = Rt$ (homogeneous Poisson immigration), then $C_{ij}(t) \sim (b/2)Cv_i v_j Rt^2$. This result was first proven by Sevastyanov (1957) in the single-type case.

**Theorem 2.** Let $W(t) = \mathbf{P}\{\mathbf{Y}(t) \neq \mathbf{0}\}$ and assume Basic conditions with $r(t) = L(t)t^{\theta}$, $\theta \in \mathbb{R}$ (in the cases **(i)-(iv)).**
**(i)** *If* $\theta > 0$ *or* $\theta = 0$ *but* $L(t)\log t \to \infty$, *then* $\lim_{t\to\infty} W(t) = 1$.
**(ii)** *If* $\theta = 0$ *and additionally* $(2C/b)L(t)\log t \to \varkappa \in (0, \infty)$, *then* $\lim_{t\to\infty} W(t) = 1 - e^{-\varkappa} \in (0, 1)$.
**(iii)** *If* $\theta \in (-1, 0)$ *or* $\theta = 0$ *but* $L(t)\log t \to 0$, *then* $W(t) \sim (2C/b)t^{\theta}L(t)\log t \to 0$, $t \to \infty$.
**(iv)** *If* $\theta = -1$, *then* $W(t) \sim (2C/b)L_1(t)t^{-1} \to 0, \quad t \to \infty$, *where* $L_1(t)$ *is a s.v.f. and* $L_1(t) = L(t)\log t + \int_0^t L(x)x^{-1}dx$.
**(v)** If $R = \int_0^{\infty} r(x)dx < \infty$ and $r(t) = o([t\log t]^{-1})$ as $t \to \infty$, then $W(t) \sim (2CR/bt)$.
**Remark 2.** In fact, $W(t) \to 1$ if $r(t)\log t \to \infty$, $W(t) \to W^* \in (0, 1)$ if $r(t)\log t \to C^* \in (0, \infty)$ and $W(t) \to 0$ if $r(t)\log t \to 0$ (cases **(iii)-(v)**).

**Theorem 3.** Assume Basic conditions ($r(t) = L(t)t^{\theta}$, $\theta \in \mathbb{R}$). *Let* $\eta_k(t) = Y_k(t)/\mathbf{E}Y_k(t)$, $k = 1, 2, \ldots, d$.
**(i)** *If* $\theta > 0$ *or* $\theta = 0$ *with* $L(t) \to \infty$ *as* $t \to \infty$*, then* $\eta_k(t) \to 1$ *in probability (**Law of Large Numbers**);*
**(ii)** *If* $\theta > 1$ *or* $\theta = 1$ *with* $\int_0^{\infty} [xL(x)]^{-1} dx < \infty$*, then* $\eta_k(t) \to 1$ *a.s. (**Strong LLN**).*
**Remark 3.** Note that by Theorem 2 in this cases $\lim_{t \to \infty} \mathbf{P}\{\mathbf{Y}(t) \neq \mathbf{0}\} = 1$.

We will use the following notation for the limiting distributions: $D(\mathbf{x}) = \mathbf{P}\{\xi_1 \leq x_1, \ldots, \xi_d \leq x_d\}, \mathbf{x} = (x_1, \ldots, x_d)$, where $\xi_1 = \ldots = \xi_d$ a.s. In fact, $D(\mathbf{x}) = \mathbf{P}\{\xi_1 \leq \min(x_1, ..., x_d)\}$.

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**Theorem 4 (CLT).** Assume Basic conditions ($r(t) = L(t)t^\theta$, $\theta \in \mathbb{R}$). *Let* $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))$, *where* $X_k(t) = [Y_k(t) - \mathbf{E}\, Y_k(t)]/\sqrt{\mathbf{Var}\, Y_k(t)}$ *for every* $k = 1, 2, \ldots, d$. *If* $\theta > 0$ *or* $\theta = 0$ *with* $L(t) \to \infty$ *as* $t \to \infty$, *then* $\lim_{t\to\infty} \mathbf{P}\{\mathbf{X}(t) \leq \mathbf{x}\} = D(\mathbf{x})$, where $\xi_1 \in N(0, 1)$.

**Corollary 2.** *From Theorem 4 and Theorem 1 one obtains the following asymptotic normality as* $t \to \infty$

$$\frac{Y_k(t)}{Cv_k(\theta+1)^{-1}L(t)t^{\theta+1}} \sim N\left(1, \frac{b(\theta+1)}{C(\theta+2)L(t)t^\theta}\right), \; k = 1, \ldots, d.$$

In the case $\theta = 0$ we have $Y_k(t)/Cv_k L(t)t \sim N(1, b/CL(t))$.

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**Theorem 5.** Assume Basic conditions with $\theta = 0$ and $r(t) = L(t) - s.v.f.$ as $t \to \infty$.

**(i)** *If* $L(t) \to r^* \in (0, \infty)$ *as* $t \to \infty$, *then*

$$\lim_{t\to\infty} \mathbf{P}\{Y_1(t)v_1^{-1}/(r^*t) \leq x_1, \dots, Y_d(t)v_d^{-1}/(r^*t) \leq x_d\} = D(\mathbf{x}),$$

*where*

$\mathbf{P}\{\xi_1 \leq x_1\} = [\beta^\alpha \Gamma(\alpha)]^{-1} \int_0^{x_1} y^{\alpha-1} e^{-y/\beta} dy, x_1 \geq 0,$

$\alpha = 2C/b$ *and* $\beta = b/2r^*$;

**(ii)** *If* $L(t) \to 0$ *but* $L(t)\log t \to \infty$ *as* $t \to \infty$, then

$$\mathbf{P}\left\{-\frac{2CL(t)}{b} \log \frac{2Y_1(t)v_1^{-1}}{bt} \leq x_1, \dots, -\frac{2CL(t)}{b} \log \frac{2Y_d(t)v_d^{-1}}{bt} \leq x_d\right\} \to$$

$\to D(\mathbf{x}),$

where $\mathbf{P}\{\xi_1 \leq x_1\} = 1 - e^{-x_1}, x_1 \geq 0.$

**(iii)** *If* $(2C/b)L(t)\log t \to \varkappa \in (0,\infty)$ *as* $t \to \infty$*, then:*

*(1) Unconditional limiting distribution*

$$\mathbf{P}\left\{\frac{\log[Y_1(t)v_1^{-1}]}{\log t} \le x_1, \dots, \frac{\log[Y_d(t)v_d^{-1}]}{\log t} \le x_d\right\} \to D(\mathbf{x}),$$

*where* $E_1(x_1) = \mathbf{P}\{\xi_1 \le x_1\} = e^{-\varkappa(1-x_1)}$, $0 \le x_1 \le 1$;

*(2) Conditional limiting distribution*

$$\mathbf{P}\left\{1 - \frac{\log[Y_1(t)v_1^{-1}]}{\log t} \le x_1, \dots, 1 - \frac{\log[Y_d(t)v_d^{-1}]}{\log t} \le x_d | \mathbf{Y}(t) \neq \mathbf{0}\right\} \to$$

$\to D(\mathbf{x})$,

*where* $E_2(x_1) = \mathbf{P}\{\xi_1 \le x_1\} = (1 - e^{-\varkappa x_1})/(1 - e^{-\varkappa})$, $0 \le x_1 \le 1$.

**Remark 4.** Remember that by Theorem 2 $\lim_{t\to\infty} \mathbf{P}\{\mathbf{Y}(t) \neq \mathbf{0}) = 1 - e^{-\varkappa} \in (0,1)$ in the case **(iii)**, while in the cases **(i)** and **(ii)** $\lim_{t\to\infty} \mathbf{P}\{\mathbf{Y}(t) \neq \mathbf{0}) = 1$.

**Theorem 6.** Assume Basic conditions ($r(t) = L(t)t^\theta$, $\theta \in \mathbb{R}$). *If* $\theta \in (-1, 0)$ *or* $\theta = 0$ *with* $L(t) = o(1/\log t)$ *then as* $t \to \infty$

$$\mathbf{P}\left\{\frac{\log[Y_1(t)v_1^{-1}]}{\log t} \le x_1, \ldots, \frac{\log[Y_d(t)v_d^{-1}]}{\log t} \le x_d | \mathbf{Y}(t) \ne \mathbf{0}\right\} \to D(\mathbf{x}),$$

*where* $\xi_1 \in U(0, 1)$.

**Remark 5.** Recall that by Theorem 2 in this case $\mathbf{P}\{\mathbf{Y}(t) \ne \mathbf{0}) \sim (2C/b)t^\theta L(t) \log t \to 0$, $t \to \infty$.

**Theorem 7.** *Assume Basic conditions with* $\theta = -1$, i.e. $r(t) = L(t)t^{-1}$. Let $\widetilde{L}(t) = \int_0^t (L(x)/x)dx$ *and* $\widehat{L}(t) = L(t)\log t$. *If* $\widetilde{L}(t)/\widehat{L}(t) \to q \in (0,\infty)$ *as* $t \to \infty$, *then*

$$\mathbf{P}\left\{\frac{\log[Y_1(t)v_1^{-1}]}{\log t} \le x_1, \ldots, \frac{\log[Y_d(t)v_d^{-1}]}{\log t} \le x_d | \mathbf{Y}(t) \neq \mathbf{0}\right\} \to D(\mathbf{x}),$$

*where* $H_1(x_1) = P(\xi_1 \le x_1) = \frac{x_1}{1+q} 1_{\{0 \le y \le 1\}} + \frac{1}{1+q} 1_{\{x_1 \ge 1\}}$ *and*

$$\mathbf{P}\left\{\frac{2Y_1(t)v_1^{-1}}{bt} \le x_1, \ldots, \frac{2Y_d(t)v_d^{-1}}{bt} \le x_d | \mathbf{Y}(t) \neq \mathbf{0}\right\} \to D(\mathbf{x}),$$

*where* $H_2(x_1) = \mathbf{P}\{\xi_1 \le x_1\} = \frac{1}{1+q} + \frac{q}{1+q}(1 - e^{-x_1}), x_1 \ge 0.$

**Remark 6.** *We obtain with different normalizations two singular limiting distributions. The non-extinction sample paths can be separated in two groups with distinct growth patterns:* ***(i)*** *with probability* $\frac{1}{1+q}$ *the growth is parabolic with a power that follows a uniform distribution on the unit interval;* ***(ii)*** *with probability* $\frac{q}{1+q}$ *the* growth is linear with an exponentially distributed slope.

**Remark 7.** *If* $\widetilde{L}(t) = \int_0^t (L(x)/x)dx \sim L(t)\log t = \widehat{L}(t)$ *then* $q = 1$ *(for example, if* $\lim_{t\to\infty} L(t) = K \in (0, \infty)$*).*
*If* $L(t) = (\log t)^\alpha$ *then* $\widehat{L}(t) = (\log t)^{\alpha+1}$*, but* $\widetilde{L}(t) = \int_K^t x^{-1}(\log x)^\alpha dx = (\alpha+1)^{-1}[(\log t)^{\alpha+1} - (\log K)^{\alpha+1}]$ *for* $\alpha \neq -1$ *and* $\widetilde{L}(t) \sim \log\log t$ *for* $\alpha = -1$*. Hence,* $q = 1/(\alpha+1)$ *if* $\alpha > -1$*,* $q = \infty$ *if* $\alpha = -1$*, and* $q = 0$ *if* $\alpha < -1$*. In the case* $\alpha < -1$ *the marginal distributions are uniformly distributed on the unit interval* $H_1(x_1) = x_1 \in (0, 1)$*, whereas in the case* $\alpha = -1$ *we have exponential distributions* $H_2(x_1) = 1 - e^{-x_1}, x_1 \geq 0$.

**Theorem 8.** *If* $R = \int_0^\infty r(x)dx < \infty$ *and* $r(t) = o([t \log t]^{-1})$, *then as* $t \to \infty$,

$$\mathbf{P}\{2Y_1(t)v_1^{-1}/bt \ \le x_1, \ldots, 2Y_d(t)v_d^{-1}/bt) \ \le x_d | \mathbf{Y}(t) \neq \mathbf{0}\} \to D(\mathbf{x}),$$

*where* $\mathbf{P}\{\xi_1 \le x\} = 1 - e^{-x}, x \ge 0.$

**Remark 8.** *The obtained limiting distribution is similar to that established for the Markov branching process without immigration. Recall that by Theorem 2 we have* $\mathbf{P}\{\mathbf{Y}(t) \neq \mathbf{0}\} \sim (2CR/bt, t \to \infty.$

One can interpret the local intensity $r(t)$ as a control function, the asymptotic behaviour of which determines different types of limiting results.

For example, some conditions are shown under which the obtained limiting results for critical MMBPwNHPI are similar to those proved for Markov branching processes without immigration, or on the other hand, similar to the processes with homogeneous Poisson immigration. New effects are discovered due to inhomogeneity: LLN and CLT, and new conditional or unconditional limiting distributions are also obtained.

# 25. Open problems

To investigate the asymptotic behaviour of MMBPwNHPI in the subcritical and supercritical cases.
To investigate Multitype age-dependent BP arising by Poisson random measures.

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# 26. References-1


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The slogan of the First World Congress of Branching Processes, Varna, Bulgaria, 1993. Only 25 years ago!

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*C. C. Heyde (Editor). Branching Processes. Proceedings of the First Word Congress. Lecture Notes in Statistics, V. 99, 1995, Springer, New York.*

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**THANK YOU VERY MUCH TO THE ORGANIZERS FOR THIS VERY ATTRACTIVE REALLY WORLD WORKSHOP!**

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**Proposition: The next Workshop to be the Second World Congress of Branching Processes!**