Properties of multitype critical Bellman–Harris processes having life-length tails of different orders

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Definitions

Consider a critical Bellman-Harris branching process (CBHP) $\mathbf{Z}^i(t) = (Z_1^i(t), \ldots, Z_n^i(t))$ with n types of particles initiated at time zero by a single particle of type $i \in \{1, 2, \ldots, n\} = \mathcal{I}$.

A type s particle of the process has life-length with distribution function $G_s(t)$ and produces at the end of its life a random number of children of different types specified by the vector $\boldsymbol{\xi}_s := (\xi_{s1}, \xi_{s2}, \ldots, \xi_{sn})$. The newborn particles of type $j \in \mathcal{I}$ evolve independently of each other and of the behaviour of the other particles, and their evolution is stochastically equivalent to the evolution of the particles of type $j \in \mathcal{I}$ described above. Define $m_{ij} := \mathbb{E}\xi_{ij}$, $\mathbf{M} := \left(m_{ij}\right)_{i,j\in\mathcal{I}}$, $\mathbf{M}(t) := \left(m_{ij}G_i(t)\right)_{i,j\in\mathcal{I}}$, $b_{jk}^s := \mathbb{E}\xi_{sj}\xi_{sk}$, $\mathbb{E}Z_j^i(t) =: P_{ij}(t)$, $q_i(t) := \mathbb{P}(\mathbf{Z}^i(t) \neq \mathbf{0})$, $q_{ij}(t) := \mathbb{P}(Z_j^i(t) \neq 0)$, $\mathbf{q}_j(t) := (q_{1j}(t), \cdots, q_{nj}(t))$, for $j, k, s \in \mathcal{I}$.

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Historical reference.

The Bellman-Harris processes were determined in the article
Bellman R., Harris T. Proc. Nat. Acad. Sci. USA, 1948, v. 34, No 12, p. 601–604.

The fundamental condition $1 - G_s(t) \neq o(t^{-2})$ for obtaining new theorems different from the Markov case was introduced in
Goldstein M.I. Critical age-dependent branching processes: Single and multitype. Z. Wahrscheinlichkeitstheor. Verw. Geb. 17, 74–88 (1971).

The next 20 years is a stormy study of these processes. The cycle of works of interest to us summarised in Vatutin's doctor dissertation "Critical branching processes with regular varying generating functions" (1987).

Main conditions for CBHP (offspring number)

$\mathcal{B}_1$ conditions

Assume that $\mathbf{Z}^i(t)$ is an irreducible, nonperiodic, critical Bellman-Harris branching process (CBHP).
It means, in particular, that $\mathbf{M}^l > \mathbf{0}$ for some integer l, the Perron root of $\mathbf{M}$ is equal to 1 and there exist unique left and right eigenvectors $\mathbf{v}$ and $\mathbf{u}$ such that

$$\mathbf{M}\mathbf{u}^T = \mathbf{u}^T,\ \mathbf{v}\mathbf{M} = \mathbf{v},\ \mathbf{v}\mathbf{u}^T = 1,\ \mathbf{u} > \mathbf{0},\ \mathbf{v} > \mathbf{0},\ \mathbf{v}\mathbf{1} = 1.$$

$\mathcal{B}_1^+$ conditions

Assume that the conditions $\mathcal{B}_1$ are true and

$$B := \frac{1}{2} \sum_{i,j,k \in \mathcal{I}} v_i b^i_{jk} u_j u_k < \infty.$$

Main conditions for CBHP (particles life-length)

$\mathcal{G}$ conditions

Set $G_j(t) =: 1 - \mathfrak{q}_j(t)$ and $\mu_j := \mathbb{E}\tau_j \le \infty$, for $j \in \mathcal{I}$, and nonlattice simultaneously. Suppose that there exists $0 \le n_0 < n$ such that

- if $j \in \{n_0 + 1, \ldots, n\} =: \mathcal{I}_1$ then $\mu_j = \infty$ and
$$\mathfrak{q}_j(t) = 1 - G_j(t) = t^{-\beta_j}\ell_j(t),$$
where $\beta_j \in (0, 1]$, $\ell_j(t)$ is a function slowly varying at infinity and $\mathfrak{q}_j(t) = O(\mathfrak{q}_n(t)) \implies \beta_n \le \beta_j$;
- if $j \in \{1, \ldots, n_0\} =: \mathcal{I}_0$ then $\mu_j < \infty$ and, additionally, $\mathfrak{q}_j(t) = o\big(\mathfrak{q}_n(t)\big)$ if $\beta_n = 1$.

Basic functions to describe the asymptotics of the moments:

$$\mu_j(t) := \int_0^t \mathfrak{q}_j(u)\mathrm{d}u \sim \begin{cases} \mu_j & \text{for } j \in \mathcal{I}_0, \\ (1-\beta_j)^{-1} t^{-\beta_j+1}\ell_j(t) & \text{for } j \in \mathcal{I}_1,\ \beta_j < 1, \\ \ell_{1j}(t) & \text{for } \beta_j = 1. \end{cases}$$

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The interest in two-type Bellman–Harris branching processes of such sort was initially motivated by the so-called branching random walks in catalytic media (S. Albeverio, L.V. Bogachev and E.B. Yarovaya, 1998–2000). We have demonstrated in 2003 that many characteristics of the branching random walks in catalytic media can be expressed in terms of CBHP. Such an approach allows us to employ known results for the critical two-type branching processes with particles whose life-length distributions have heavy tails and to analyse. In these models $\mathfrak{q}_1(t) = e^{-ct}$ and $\mathfrak{q}_2(t) \sim Ct^{-\beta_2}$ for $\beta_2 = 1/2, 1$. Analogical results has been further obtained (2013 VT) for presented model with $n = 2$. But in the case $n = 2$ there are a lot of opened problems.

Main conditions for CBHP

$\mathcal{G}_1$ conditions (Sufficient conditions for regularity of increments for renewal matrix)

Suppose that for CBHP

- the conditions $\mathcal{G}$ and $\mathcal{B}_1$ are true;
- if $\beta_n \in (0, 0.5]$ then there exists a constant $C > 0$ such that

$$G_j(t+\Delta) - G_j(t) \leq \Delta C t^{-1-\beta_n} \ell_n(t)$$

for all $j \in \mathcal{I}$, $t > 1$ and all fixed $\Delta > 0$.

First moments asymptotics

Theorem (Vatutin & Topchii, 2017)

Let for CBHP $\mathbf{Z}^i(t)$ the $\mathcal{G}_1$ conditions are true. Then

$$\mathbf{P}(t) = \mathbf{D}\mathfrak{D}(t)(1+o(1)), \text{ previous to } \left[P_{ij}(t) = C_{ij} + o(1)\right]$$

for some matrix $\mathbf{D} > \mathbf{0}$ and diagonal matrix $\mathfrak{D}(t)$ with elements

$$\mathfrak{d}_j(t) := \mathbb{C}(\beta_j, \beta_n)\mu_j(t)\mu_n^{-1}(t),$$

$$\mu_j(t)\mu_n^{-1}(t) \sim \begin{cases} \mu_j t^{\beta_n - 1}\ell_n^{-1}(t), & \text{as } \beta_n \neq 1, j \in \mathcal{I}_0, \\ \mu_j \ell_{1n}^{-1}(t), & \text{as } \beta_n = 1, j \in \mathcal{I}_0, \\ t^{\beta_n - \beta_j}\ell_n^{-1}(t)\ell_j(t), & \text{as } \beta_n, \beta_j \neq 1, j \in \mathcal{I}_1, \\ t^{\beta_n - 1}\ell_n^{-1}(t)\ell_{1j}(t), & \text{as } \beta_n < \beta_j = 1, j \in \mathcal{I}_1, \\ \ell_{1n}^{-1}(t)\ell_{1j}(t), & \text{as } \beta_n = \beta_j = 1, j \in \mathcal{I}_1. \end{cases}$$

Prehistory Theorem (Vatutin & Topchii, 2017)

From the main part of published results about the multitype CBHP it is easy to obtain $\mathfrak{d}_j(t) = c_j + o(1)$ if $\mathfrak{q}_j(t) = (c_j + o(1))\mathfrak{q}_n(t)$.

In the case $n = 2$, $\mathcal{I}_0 = \{1\}$ and $\mathcal{I}_1 = \{2\}$ for all $\beta_2 \in (0, 1]$ the previous result was obtained by Vatutin & Topchii (2013). Earlier such sort of results was proofed for $\beta_2 = 0, 0.5$ Vatutin, Topchii, Yarovaya (2003), 1 as applications to random branching walks.

The analogy results was obtained for $\mathbb{E}\big(Z_j^i(t)\big)^2$.

Second moments asymptotics

Theorem (Vatutin & Topchii, 2017 (1))

Let the conditions of Theorem (Vatutin & Topchii, 2017) and $\mathcal{B}_1^+$ are true and for $j \in \mathcal{I}_1$ in the case $\beta_j > 1/2$

$$G_j(t+\Delta) - G_j(t) = o(t^{-1}).$$

Set $\mathcal{I}^0 := \left\{ j : \int_0^\infty \mathfrak{d}_j^2(t)\mathrm{d}t < \infty \right\} = \left\{ j : \int_0^\infty \mu_j^2(t)\mu_n^{-2}(t)\mathrm{d}t < \infty \right\}$.

Then

$$\mathbb{E}\left(Z_j^i(t)\right)^2 \sim \begin{cases} \mathbb{S}_{ij}\mu_n^{-1}(t)\int_0^t \mathfrak{d}_j^2(u)\mathrm{d}u, & \text{as } \; j \in \mathcal{I} \setminus \mathcal{I}^0, \\ \mathbb{S}_{ij}\mu_n^{-1}(t) + C_{ij}\mathfrak{d}_j(u), & \text{as } \; j \in \mathcal{I}^0, \end{cases} \tag{1}$$

for some constants $\mathbb{S}_{ij} = \mathbb{S}_{ij}(G_i(\cdot), G_j(\cdot))$ and C_{ij}. Previous was $\mathbb{E}\left(Z_j^i(t)\right)^2 = (C_{ij}c_j + o(1))t\mu_n^{-1}(t)$ if $\mathfrak{q}_j(t) = (c_j + o(1))\mathfrak{q}_n(t)$.

Prehistory

Theorem (Vatutin (1979))

Let $\mathbf{Z}^i(t)$ for critical Bellman-Harris process, the conditions $\mathcal{B}_1^+$ are true, $1 - G_j(t) = \mathfrak{q}_j(t) = (c_j + o(1))\mathfrak{q}_n(t)$ for $j \in \mathcal{I}$ and $\mathbf{c} = \big(c_1, \ldots, c_n\big)$, then $q_i(t) = \mathbb{P}\big\{\mathbf{Z}^i(t) \neq \mathbf{0}\big\} \sim u_i\sqrt{\mathfrak{q}_n(t)B^{-1}}$ and for all $\mathbf{0} \leq \mathbf{s} \leq \mathbf{1}$ there are exists independent of i limit

$$\lim_{t\to\infty} \mathbb{E}\big\{\mathbf{s}^{\mathbf{Z}^i(t)}\big|\mathbf{Z}^i(t) \neq \mathbf{0}\big\} = 1 - \sqrt{\frac{(\mathbf{v}\otimes\mathbf{c}, \mathbf{1} - \mathbf{s})}{(\mathbf{v}, \mathbf{l})}}.$$

In particular ($c_j = 0$ – was an unsolved problem)

$$q_{ij}(t) = \mathbb{P}\big\{Z_j^i(t) \neq 0\big\} = c_{0i}\sqrt{\mathfrak{q}_n(t)}\big(\sqrt{c_j} + o(1)\big).$$

Main representation

For $\mathbf{s} := (s_1, \dots, s_n) \in [0,1]^n$ and $\mathbf{z} := (z_1, \dots, z_n) \in \mathbb{Z}_+^n$ define $\mathbf{s}^{\mathbf{z}} := s_1^{z_1} \cdot \dots \cdot s_n^{z_n}$ and denote

$$F_i(t;\mathbf{s}) = F_i(t;s_1,\dots,s_n) := \mathbb{E}\mathbf{s}^{\mathbf{Z}^i(t)}, \ \mathbf{F}(t;\mathbf{s}) := (F_1(t;\mathbf{s}),\dots,F_n(t;\mathbf{s}))^T.$$

For vectors $\mathbf{x} = (x_1, \cdots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$ put $\mathbf{x} \otimes \mathbf{y} : =(x_1 y_1, \dots, x_n y_n)^T$. Introduce column-vectors $\mathbf{G}(t) := \left(G_1(t), \dots, G_n(t)\right)^T$, $\mathbf{f}(\mathbf{s}) := \left(\mathbb{E}\mathbf{s}^{\boldsymbol{\xi}_1}, \dots, \mathbb{E}\mathbf{s}^{\boldsymbol{\xi}_n}\right)^T$.

Standard total probability formula arguments leads to the system of integral equations (see B.A. Sevastyanov, Branching Processes, 1971) $\left[\mathbf{1} - \mathbf{F}(t;\mathbf{s}(\mathbf{t}))\right.$ all limit theorems$\left.\right]$ main representation

$$\mathbf{F}(t;\mathbf{s}) = \mathbf{s} \otimes (\mathbf{1} - \mathbf{G}(t)) + \int_0^t \mathbf{f}(\mathbf{F}(t-w,\mathbf{s})) \otimes d\mathbf{G}(w).$$

Representation for moments

Set $\mathbf{G_I}(t) := (G_i(t)\delta_{ij})_{i,j=1}^n$ and $\mathbf{I}(t) = \mathbf{I}1_{\{t\geq 0\}} = \big(\delta_{ij}(t)\big)_{i,j=1}^n 1_{\{t\geq 0\}}$,

$$P_{ij}(t) := \mathbb{E}Z_j^i(t) = \frac{\partial F_i(t,\mathbf{s})}{\partial s_j}\Big|_{\mathbf{s}=\mathbf{1}},\ \mathbf{P}(t) := \big(P_{ij}(t)\big)_{i,j=1}^n.$$

Differentiating both parts of the main representation with respect to s_j at $\mathbf{s} = \mathbf{1}$ we conclude that (matrix renewal equation)

$$\mathbf{P}(t) = \mathbf{I}(t) - \mathbf{G_I}(t) + \int_0^t d\mathbf{M}(u)\mathbf{P}(t-u),\ \text{or}\ \mathbf{P}(t) = \mathbf{U}*(\mathbf{I}(\cdot) - \mathbf{G_I}(\cdot))(t),$$

were $\mathbf{U}(t) := \sum_{k=0}^\infty \mathbf{M}^{*k}(t)$ $\left(\mathbf{U}(t) = \mathbf{I}(t) + \int_0^t d\mathbf{M}(u)\mathbf{U}(t-u)\right)$ – renewal matrix. Free term.

The asymptotic behaviour of the renewal matrix $\mathbf{U}(t)$

Set $\mathbf{\Pi}(t) = \mathbf{F}(t) + \mathbf{M} * \mathbf{\Pi}(t)$, then $\mathbf{\Pi}(t) = \mathbf{U} * \mathbf{F}(t)$ were $\mathbf{U}(t) = \sum_{k=0}^{\infty} \mathbf{M}^{*k}(t)$ is the renewal matrix.

Topchii 2017 Mathematical Proceedings (Siberian Advances in Mathematics)

If $\mathfrak{q}_j(t) = (c_j + o(1))\mathfrak{q}_n(t) = (c_j + o(1))t^{-\beta_n}\ell_n(t)$ for some $c_j \geq 0$, then under $\mathcal{B}_1$ conditions for some $\mathbf{D}_{\beta_n} > \mathbf{0}$

$$\mathbf{U}(t) \sim \mathbf{D}_{\beta_n} t^{\beta_n} \ell_n^{-1}(t) \sim \mathbf{D}_{\beta_n} t \mu_n^{-1}(t).$$

Under $\mathcal{G}_1$ conditions for all fixed $\Delta > 0$

$$\mathbf{U}(t+\Delta) - \mathbf{U}(t) \sim \Delta \beta_n \mathbf{D}_{\beta_n} \mu_n^{-1}(t).$$

Increments of renewal matrix in infinite mean case

One-dimensional case: Teugels J.L. (1968); Garsia A., Lamperti J. (1962/63); Erickson K.B. (1970-71); Doney R.A. (1997)
Two-dimensional case:

Theorem (Vatutin, Topchij (2013))

Let $\mathcal{B}_1$ and $\mathcal{G}$ conditions are true and there exist positive constants C and T_0 such that for $t \geq T_0$ and any fixed $\Delta > 0$

$$G_2(t+\Delta) - G_2(t) \leq C\Delta\mu_2^{-1}(t).$$

if $\beta_2 \in (0, 1/2]$ [$\mathcal{G}_1$ conditions]. Then

$$\mathbf{U}(t+\Delta) - \mathbf{U}(t) \sim \Delta\beta_2\mathbf{D}_{\beta_2}\mu_2^{-1}(t).$$

Renewal equation for generating functions

Let $\boldsymbol{\phi}(\mathbf{s}) = \left(\phi_1(\mathbf{s}), \phi_2(\mathbf{s}), \ldots, \phi_n(\mathbf{s})\right) := \mathbf{M}\mathbf{s} - \left(\mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{s})\right)$ and $N_i(\mathbf{x}) = \frac{1}{2}\sum_{j,k\in\mathcal{I}} b^i_{jk} x_j x_k$, $\mathbf{N}(\mathbf{x}) = \left(N_1(\mathbf{x}), N_2(\mathbf{x}), \ldots, N_n(\mathbf{x})\right)$ for $\mathbf{x} := (x_1, \ldots, x_n) \in [0,1]^n$ and $i \in \mathcal{I}$. If $s_j \to 1 - 0$ for $j \in \mathcal{I}$, then

$$\boldsymbol{\phi}(\mathbf{s}) = \mathbf{M}\mathbf{s} - (\mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{s})) = \mathbf{N}(\mathbf{1} - \mathbf{s}) + o\left(\mathbf{1}\left\|\mathbf{1} - \mathbf{s}\right\|^2\right).$$

Define $\mathbf{Q}(t;\mathbf{s}) := \mathbf{1} - \mathbf{F}(t;\mathbf{s})$ and rewrite the main representation in the forms

$$\mathbf{Q}(t;\mathbf{s}) = (\mathbf{1} - \mathbf{s}) \otimes (\mathbf{1} - \mathbf{G}(t)) + \int_0^t \left(\mathbf{1} - \mathbf{f}(\mathbf{F}(t - w;\mathbf{s}))\right) \otimes \mathrm{d}\mathbf{G}(w),$$

$$\mathbf{Q}(t;\mathbf{s}) = (\mathbf{1} - \mathbf{s}) \otimes (\mathbf{1} - \mathbf{G}(t)) - \mathbf{G}^{(q)}(t;\mathbf{s}) + \int_0^t \mathrm{d}\mathbf{M}(w)\mathbf{Q}(t - w;\mathbf{s}),$$

where $\mathbf{G}^{(q)}(t;\mathbf{s}) := \mathbf{G}_{\mathbf{I}} * \boldsymbol{\phi}(\mathbf{Q}(\cdot;\mathbf{s}))(t)$.

Renewal equation for generating functions

The last version the main representation for $\mathbf{Q}(t;\mathbf{s}) := \mathbf{1} - \mathbf{F}(t;\mathbf{s})$ is the renewal equation with solution

$$\mathbf{Q}(t;\mathbf{s}) = \mathbf{U} * (\mathbf{1} - \mathbf{s}) \otimes (\mathbf{1} - \mathbf{G}(\cdot))(t) - \mathbf{U} * \mathbf{G}_{\mathbf{I}} * \phi(\mathbf{Q}(\cdot;\mathbf{s}))(t).$$

Set $\mathbf{s}_j = (\delta_{ij})_{i=1,\cdots,n}$, then $Q_i(t;\mathbf{1} - \mathbf{s}_j) = q_{ij}(t)$, $\phi_{ij}(t) := \phi_i(\mathbf{q}_j(t))$. The description of the probability of the presence of particles was reduced to investigating the asymptotics of the solution of the equations system

$$q_{ij}(t) = P_{ij}(t) - \sum_{k \in \mathcal{I}} G_k * \phi_{kj} * U_{ik}(t),$$

where $\phi_{kj}(t) \asymp q_{ij}^2(t)$. Note that $q_{ij}(t) \leq P_{ij}(t)$.

Define $\Phi_{kj}(t) = \int_0^t \phi_{kj}(u)du$, $\Phi_{kj} := \Phi_{kj}(\infty) \le \infty$.
Recall that $j \in \mathcal{I}^0$ if $\int_0^\infty \mu_j^2(u)\mu_n^{-2}(u)du < \infty$.

Theorem (Case $j \in \mathcal{I}^0$)

Under the conditions of Theorem Vatutin, Topchij (2017) (asymptotic of $P_{ij}(t)$) for $j \in \mathcal{I}^0$

$$q_{ij}(t) \sim c_{ij} P_{ij}(t).$$

Then

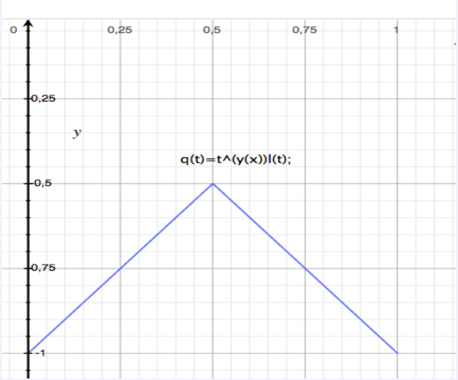
$$q_{ij}(t) \sim c_{ij} \begin{cases} \mu_n^{-1}(t), \ j \in \mathcal{I}_0, \\ \frac{\mu_j(t)}{\mu_n(t)}, \ j \in \mathcal{I}_1 \end{cases} \sim c_{ij} \begin{cases} t^{\beta_n - 1} \ell_n^{-1}(t), \ \beta_n \in (0, 0.5], \\ t^{\beta_n - \beta_j} \frac{\ell_j(t)}{\ell_n(t)}, \ \beta_j - \beta_n \in [0.5, 1). \end{cases}$$

If $j \in \mathcal{I}_0$, $\beta_n = 0.5$ and $j \notin \mathcal{I}^0$, then

$$q_{ij}(t) \sim t^{-0.5} \ell(t) \sim \mu_n^{-1}(t) \Big/ \int_0^t \mathfrak{q}_n^2(u)du.$$

Example of BHP

Set $j \in \mathcal{I}_0$. For $\beta_n > 0.5$ will be true $q_{ij}(t) \sim \mathfrak{q}_n(t)$ very likely hypothesis. We combine Theorem and the hypothesis in the relation $q_{ij}(t) \sim t^{y(\beta_n)} l(t)$, then the graph of the function $y = y(x) = y(\beta_n)$ has the form



Hypoceses

If $j \in \mathcal{I}_1$, $\beta_j - \beta_n \leq 0.5$ and $j \notin \mathcal{I}^0$, then under additional conditions of the type $\mathfrak{q}_s(t) = o(t^{-3}\mathfrak{q}_n(t))$ for $s \in \mathcal{I}_0$ and $\mathfrak{q}_s'''(t)$ are regularly varying for $s \in \mathcal{I}_1$ it will be

$$P_{ij}(t) \sim \sum_{k \in \mathcal{I}} G_k * \phi_{kj} * U_{ik}(t).$$

It is plausible that the last is possible if conditions

$$q_{ij}(t) \sim c_{ij}\sqrt{\mathfrak{q}_j(t)} \asymp t^{-\beta_j/2}$$

are satisfied.

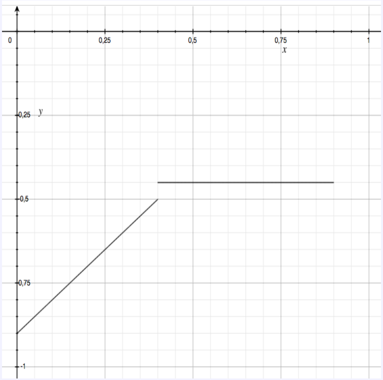
In the case $\beta_j \leq 0.5$, this result seems to be natural.

But for $j \in \mathcal{I}_1$, $\beta_j - \beta_n \geq 0.5$ and $j \in \mathcal{I}^0$ we have

$$q_{ij}(t) \asymp t^{\beta_n - \beta_j}.$$

Example of BHP

Set $j \in \mathcal{I}_1$. Recall that $\beta_n \leq \beta_j$. For $\beta_j = 0.9 > 0.5$ we combine the theorem and the hypothesis in the relation $q_{ij}(t) \sim t^{y(\beta_j, \beta_n)} l(t)$, then the graph of the function $y = y(x) = y(\beta_j, \beta_n)$ has the form



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