Extinction rate of continuous state branching processes in critical Lévy environments

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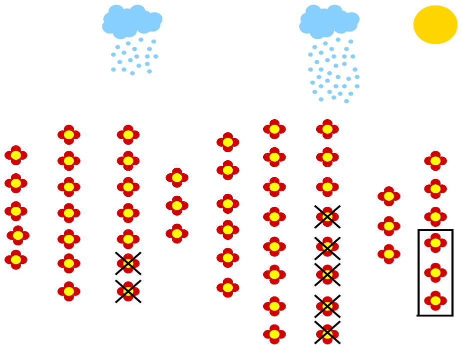
Galton-Watson processes

- Models for population dynamics in discrete time and space.
- No interaction between individuals

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i^{(n)}$$

where $(\xi_j^{(k)}, (j,k) \in \mathbb{N}^2)$ are iid with law $\mathcal{Q}$.

- Galton-Watson processes in iid random environment: reproduction law changes at each generation



Galton-Watson processes in random environment

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i^{(n)}, \qquad \text{where } (\xi_j^{(n)}, j \in \mathbb{N}) \text{ are iid with law } \mathcal{Q}_n.$$

- Define the random walk $(S_n, n \in \mathbb{N})$ by

$$X_n = \log m(\mathcal{Q}_n) \qquad S_n = \sum_{k=1}^{n} X_i$$

where

$$m(\mathcal{Q}_n) = \sum_{y=0}^{\infty} y \mathcal{Q}_n(\{y\})$$

- Then $Z_n e^{-S_n}$ is a martingale $\Longrightarrow$ Properties of $(Z_n, n \in \mathbb{N})$ determined by its associated random walk $(S_n, n \in \mathbb{N})$ (in particular subcritical/critical/supercritical).

Asymptotic behaviour of Galton-Watson processes in random environment has been well studied

- First work: Smith and Wilkinson 69, Athreya and Karlin 71
- Reviews: Birkner, Geiger and Kersting 05, Dyakonova, Vatutin and Sagitov 11

GW processes in random environment

Lévy processes and CSBP

CSBP in random environment

Main result

Lévy processes

Definition

Stochastic process issued from the origin with iid increments and almost sure right continuous paths

= CONTINUOUS VERSION OF RANDOM WALKS

Continuous State Branching Processes (CSBP)

Definition

- Non-negative strong Markov process $(Y_t, t \geq 0)$ where 0 and ∞ are two absorbing states
- Branching property: if $Y^{(x)}$ is the process with initial state x,

$$Y^{(x+y)} \stackrel{\mathcal{L}}{=} \tilde{Y}^{(x)} + \hat{Y}^{(y)},$$

where $\tilde{Y}^{(x)}$ and $\hat{Y}^{(y)}$ are independent

(Lamperti 1967)

CSBP $\Leftrightarrow$ SCALING LIMITS OF GALTON-WATSON PROCESSES

Properties

The law of Y is completely characterized by its Laplace transform

$$\mathbb{E}_x e^{-\lambda Y_t} = e^{-x u_t(\lambda)}, \qquad \forall x > 0, \; t \geq 0,$$

where u is a differentiable function in t satisfying

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)), \qquad u_0(\lambda) = \lambda.$$

ψ: branching mechanism. Given by the Lévy-Khincthine formula

$$\psi(\lambda) = \psi'(0^+)\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(dx),$$

$(\psi'(0^+), \gamma) \in \mathbb{R}^2$, μ σ-finite measure s.t. $\int (x \wedge x^2)\mu(dx) < \infty$.

Can be defined as the unique strong solution (Fu and Li 10) of the SDE

$$Y_t = Y_0 - \int_0^t \psi'(0^+) Y_s dz + \int_0^t \sqrt{2\gamma^2 Y_s} dB_s + \int_0^t \int_0^\infty \int_0^{Y_{s-}} z\tilde{N}^{(b)}(ds, dz, du),$$

where B is a standard Brownian motion, $N^{(b)}(ds, dz, du)$ is a Poisson random measure with intensity $ds\mu(dz)du$ independent of B, and $\tilde{N}^{(b)}$ is the compensated measure of N.

- CSBP in iid random environment? Catastrophes, continuous variations,...?
- Boeinghoff and Hutzenthaler 12, Palau and Pardo 15: Brownian motion, Bansaye, Pardo and S. 13: compound Poissons processes; Palau, Pardo and S. 16, Li and Xu 17: more general environment but stable branching mechanism
- A Lévy process somewhere...
- See Bansaye and Simatos 2015, Bansaye, Caballero, Méléard 2018+ for general results on scaling limits of GW processes in random environment
- See Palau and Pardo 2015 and He, Li and Xu 2016 for existence and unicity of strong solutions of SDE

Unique non-negative strong solution of (Palau and Pardo 2015, He, Li and Xu 2016)

$$Z_t = Z_0 - \int_0^t \psi'(0^+) Z_s dz + \int_0^t \sqrt{2\gamma^2 Z_s} dB_s^{(b)} + \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\tilde{N}^{(b)}(ds, dz, du) + \int_0^t Z_{s-} dK_s,$$

where K independent Lévy process

$$K_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^v - 1)\tilde{N}^{(e)}(ds, dv) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^v - 1)N^{(e)}((ds, dv),$$

$\alpha, \sigma \in \mathbb{R}$, $B^{(e)}$ Brownian motion, $N^{(e)}$ Poisson measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $ds\pi(dy)$, and π σ-finite s.t.

$$\int_{\mathbb{R}} (1 \wedge v^2)\pi(dv) < \infty.$$

We will define an auxiliary process $\overline{K}$:

$$K_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^v - 1)\tilde{N}^{(e)}(ds, dv) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^v - 1)N^{(e)}((ds, dv).$$

$$\overline{K}_t = \overbrace{\left(\alpha - \psi'(0^+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(dv)\right)}^{\tilde{\alpha}} t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} v\widetilde{N}^{(e)}(ds, dv) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(ds, dv),$$

then $Z_t e^{-\overline{K}_t}$ quenched martingale $\Longrightarrow$ Will give the long term behaviour of the process (in particular subcritical/critical/supercritical).

In the stable case

$N^{(b)} = 0$ or $\gamma^2 = 0$ and $N^{(b)}$ is a Poisson random measure with intensity $c_\beta \beta(\beta+1) dsdzdu/(\Gamma(1-\beta) z^{2+\beta})$

Proposition (Bansaye, Pardo and S. 2013, Palau and Pardo 2015, He, Li and Xu 16, Palau, Pardo and S. 16, Li and Xu 17)

For all $z, \lambda > 0$ and $t \geq 0$, we have

$$\mathbb{E}_z\left[e^{-\lambda Z_t e^{-\overline{K}_t}} \middle| \overline{K}\right] = e^{-z\left(\lambda^{-\beta} + \beta c_\beta \int_0^t e^{-\beta \overline{K}_u} du\right)^{-1/\beta}}.$$

Consequence

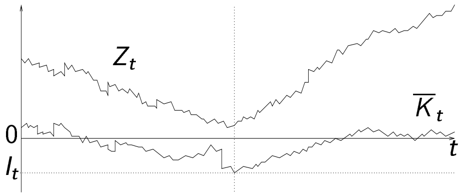
$$\lambda \to \infty \Rightarrow \mathbb{P}(Z_t e^{-\overline{K}_t} = 0) = \mathbb{P}(Z_t = 0)$$

Beyond the stable case?

No more explicit expression of the extinction probability as a function of the random environment.
We will focus on the critical case, $\overline{K}$ oscillates.

Idea (Afanasyev et al 2005)

Decomposition with respect to the minimum of the auxiliary Lévy process



GW processes in random environment

Lévy processes and CSBP

CSBP in random environment

Main result

Spitzer's condition

$$\frac{1}{t}\int_0^t \mathbf{P}(\overline{K}_t \geq 0)dt \longrightarrow \rho \in (0,1), \qquad \text{as} \qquad t \to \infty.$$

Absorption (+technical)

$$\exists C, \beta > 0, \quad \psi_0(\lambda) = (\psi(\lambda) - \lambda\psi'(0^+)) \geq C\lambda^{1+\beta} \text{ and } \mathbb{E}[e^{\beta \bar{K}}] < \infty.$$

Implies Grey's condition ($\int^\infty \frac{dz}{\psi_0(z)} < \infty$,), which is a necessary and sufficient condition for absorption (He, Li and Xu 2016)

Survival under favorable environment

$$\int^\infty \theta \ln^2(\theta)\mu(d\theta) < \infty.$$

+ Technical assumptions, to ensure the smoothness of the renewal measure V of $\overline{K}$.

Running infimum of $\bar{K}$

$$I_t = \inf_{0\leq s\leq t} \overline{K}_s, \qquad t \geq 0.$$

Theorem (Bansaye, Pardo and S. 2018+)

There exists a function c such that for any $z > 0$,

$$\mathbb{P}_z(Z_t > 0) \sim c(z)\mathbf{P}_1(I_t > 0) \sim c_0 c(z) t^{\rho-1}\ell(t), \quad \text{as} \quad t \to \infty,$$

where ℓ is a slowly varying function.

Renewal function

$$V(x) := \mathbf{E}\left[\int_{[0,\infty)} \mathbf{1}_{\{I_t \geq -x\}} d\widehat{L}_t\right],$$

where $\widehat{L}$ is the local time of the reflected process $\overline{K} - I$.

Law of $\overline{K}$ conditioned to be positive: for $\Lambda \in \mathfrak{G}_t$ (gen by $(Z, \overline{K})$) and $x > 0$:

$$\mathbb{P}^{\uparrow}_{(z,x)}(\Lambda) = \frac{1}{V(x)} \mathbb{E}_{(z,x)}\left[V(\overline{K}_t)\mathbf{1}_{\{I_t \geq 0\}}\mathbf{1}_{\Lambda}\right].$$

It allows to consider the CSBP Z in a Lévy environment conditioned to be positive

Let us introduce

$$h(x,y) = \frac{V(x+y)}{V(x)}\mathbf{1}_{\{x+y>0\}}$$

Theorem (Bansaye, Pardo and S. 2018+)

The process $((Z_t, K_t, \overline{K}_t), t \geq 0)$, under $\mathbb{P}^{\uparrow}_{(z,x)}$ for $x, z > 0$, is a strong Markov process and has the same law as $((Z^{\uparrow}_t, K^{\uparrow}_t, \overline{K}^{\uparrow}_t), t \geq 0)$ which is a weak solution of the following equations:

$$\overline{K}^{\uparrow}_t = x + \tilde{\alpha} t + \sigma B^{(c)}_t + \int_0^t \left(\sigma^2 \frac{v(\overline{K}^{\uparrow}_s)}{V(\overline{K}^{\uparrow}_s)} + \int_{(-1,1)} \Big(h(\overline{K}^{\uparrow}_s, y) - 1 \Big) y \mathbf{1}_{\{\overline{K}^{\uparrow}_s + y > 0\}} \pi(dy) \right) ds$$
$$+ \int_0^t \int_{(-1,1)} \int_0^{h(\overline{K}^{\uparrow}_{s-}, z)} z \widetilde{N}^{(c)}(ds, dz, du) + \int_0^t \int_{\mathbb{R} \setminus (-1,1)} \int_0^{h(\overline{K}^{\uparrow}_{s-}, z)} z N^{(c)}(ds, dz, du),$$

where $B^{(c)}$ is a standard Brownian motion independent of the Poisson random measure $N^{(c)}$ whose intensity is given by $ds\pi(dz)du$.

Theorem (Bansaye, Pardo and S. 2018+)

$$Z_t^{\uparrow} = z - \int_0^t \psi'(0+) Z_s^{\uparrow} ds + \int_0^t \sqrt{2\gamma^2 Z_s^{\uparrow}} dB_s^{(b)} + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}^{\uparrow}} z \widetilde{N}^{(b)}(ds, dz, du) + \int_0^t Z_{s-}^{\uparrow} dK_s^{\uparrow},$$

Theorem (Bansaye, Pardo and S. 2018+)

$$K_t^{\uparrow} = \alpha t + \sigma B_t^{(c)} + \int_0^t \left(\sigma^2 \frac{v(\overline{K}_s^{\uparrow})}{V(\overline{K}_s^{\uparrow})} + \int_{(-1,1)} \Big(h(\overline{K}_s^{\uparrow}, y) - 1 \Big) (e^y - 1) \mathbf{1}_{\{\overline{K}_s^{\uparrow} + y > 0\}} \pi(dy) \right) ds$$
$$+ \int_0^t \int_{(-1,1)} \int_0^{h(\overline{K}_{s-}^{\uparrow}, z)} (e^z - 1) \widetilde{N}^{(c)}(ds, dz, du)$$
$$+ \int_0^t \int_{\mathbb{R} \setminus (-1,1)} \int_0^{h(\overline{K}_{s-}^{\uparrow}, z)} (e^z - 1) N^{(c)}(ds, dz, du),$$

$$\overline{K}_t^{\uparrow} = x + \tilde{\alpha} t + \sigma B_t^{(c)} + \int_0^t \left(\sigma^2 \frac{v(\overline{K}_s^{\uparrow})}{V(\overline{K}_s^{\uparrow})} + \int_{(-1,1)} \big(h(\overline{K}_s^{\uparrow}, y) - 1 \big) y \mathbf{1}_{\{\overline{K}_s^{\uparrow} + y > 0\}} \pi(dy) \right) ds$$
$$+ \int_0^t \int_{(-1,1)} \int_0^{h(\overline{K}_{s-}^{\uparrow}, z)} z \widetilde{N}^{(c)}(ds, dz, du) + \int_0^t \int_{\mathbb{R} \setminus (-1,1)} \int_0^{h(\overline{K}_{s-}^{\uparrow}, z)} z N^{(c)}(ds, dz, du).$$

Proof of the main theorem

$$\mathbb{P}_z(Z_t > 0) \sim c(z)\mathbf{P}_1(I_t > 0)$$

$$\begin{aligned}\mathbb{P}_z(Z_t > 0) &= \mathbb{P}_{(z,x)}(Z_t > 0, I_t > -y) + \mathbb{P}_{(z,x)}(Z_t > 0, I_t \leq -y) \\ &= \mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0) + \mathbb{P}_{(z,x)}(Z_t > 0, I_t \leq -y)\end{aligned}$$

i) $\forall \varepsilon > 0, \exists y,$

$$\mathbb{P}_{(z,x)}(Z_t > 0, I_t \leq -y) \leq \varepsilon \mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0)$$

ii) $\mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0) \sim c(z, x+y)\mathbf{P}_1(I_t > 0)$

i) $\mathbb{P}_{(z,x)}(Z_t > 0, I_t \leq -y) \leq \varepsilon \mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0)$

If v satisfies

$$\frac{\partial}{\partial s} v_t(s,\lambda,\overline{K}) = e^{\overline{K}_s}\psi_0(v_t(s,\lambda,\overline{K})e^{-\overline{K}_s}), \qquad v_t(t,\lambda,\overline{K}) = \lambda,$$

then

$$e^{-Z_s v_t(s,\lambda,\overline{K})e^{-\overline{K}_s}},$$

is a martingale. Hence

$$\mathbb{P}_{(z,x)}(Z_t > 0|\overline{K}) = 1 - e^{-ze^{-x}v_t(0,\infty,\overline{K})} \leq ze^{-x}v_t(0,\infty,\overline{K}).$$

Assumption $\psi_0(\lambda) \geq C\lambda^{1+\beta} \Rightarrow$ for $s \leq t$ and $\lambda \geq 0$,

$$\partial_s v_t(s,\lambda,K) \geq Ce^{\overline{K}_s}\left(v_t(s,\lambda,\overline{K})e^{-\overline{K}_s}\right)^{\beta+1} = Cv_t^{\beta+1}(s,\lambda,\overline{K})e^{-\beta\overline{K}_s}.$$

This implies

$$v_t(0,\infty,\overline{K}) \leq \left(\beta C\int_0^t e^{-\beta\overline{K}_s}ds\right)^{-1/\beta}.$$

i) $\mathbb{P}_{(z,x)}(Z_t > 0, I_t \leq -y) \leq \varepsilon \mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0)$

We get the following bound

$$\mathbb{P}_{(z,x)}(Z_t > 0, I_t < -y) \leq C(z,x)\mathbf{E}_x\left[1 \wedge \left(\int_0^t e^{-\beta \overline{K}_s} ds\right)^{-1/\beta}, I_t < -y\right].$$

Following Li and Xu 17, introduce τ_{-y} hitting time of $-y$ by $\overline{K}$,

$$\mathbf{E}_x\left[1 \wedge \left(\int_0^t e^{-\beta \overline{K}_s} ds\right)^{-1/\beta}, \tau_{-y} \leq t\right]$$
$$\leq e^{-y} \sum_{i=1}^{\lfloor t \rfloor - t_0} \mathbf{E}\left[\left(\int_{\tau_{-y}}^{t+\tau_{-y}-i} e^{-\beta(\overline{K}_s - \overline{K}_{\tau_{-y}})} ds\right)^{-1/\beta}, i < \tau_{-y} \leq i+1\right]$$
$$+ \mathbf{P}(\lfloor t \rfloor - t_0 < \tau_{-y} \leq t)$$

ii) $\mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0) \sim c(z, x+y)\mathbf{P}_1(I_t > 0)$

$$\mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0) = \mathbb{P}_{(z,x+y)}(Z_t > 0 | I_t > 0)\mathbf{P}_{x+y}(I_t > 0)$$
$$\sim \mathbb{P}_z^{\uparrow}(Z_s > 0, \forall s \geq 0)\mathbf{P}_{x+y}(I_t > 0)$$

$$e^{-Z_s^{\uparrow} v_t(s,\lambda,\overline{K}^{\uparrow})e^{-\overline{K}_s^{\uparrow}}}, 0 \leq s \leq t \quad \text{supermartingale}$$

$$\mathbb{P}_z^{\uparrow}(Z_t = 0) \leq \mathbf{E}_x^{\uparrow}\left[e^{-zv_t(0,\infty,\overline{K}^{\uparrow})e^{-x}}\right] ? <?1.$$

ii) $\mathbb{P}_{(z,x+y)}(Z_t > 0, I_t > 0) \sim c(z, x+y)\mathbf{P}_1(I_t > 0)$

$$\psi_0(\lambda) = \lambda\Phi(\lambda) \quad \Rightarrow \Phi \text{ non decreasing.}$$

$$\partial_s v_t(s,\lambda,\overline{K}^{\uparrow}) = v_t(s,\lambda,\overline{K}^{\uparrow})\Phi(e^{-\overline{K}_s^{\uparrow}} v_t(s,\lambda,\overline{K}^{\uparrow})) \leq v_t(s,\lambda,\overline{K}^{\uparrow})\Phi(e^{-\overline{K}_s^{\uparrow}}\lambda)$$

which entails

$$v_t(0,\infty,\overline{K}^{\uparrow}) \geq v_t(0,\lambda,\overline{K}^{\uparrow}) \geq \lambda \exp\left\{-\int_0^t \Phi\left(\lambda e^{-\overline{K}_s^{\uparrow}}\right) ds\right\},$$

and thus

$$\mathbb{P}^{\uparrow}_{(z,x)}(Z_t = 0) \leq \mathbf{E}^{\uparrow}_x\left[e^{-ze^{-x}\lambda \exp\left\{-\int_0^t \Phi\left(\lambda e^{-\overline{K}_s}\right) ds\right\}}\right].$$

Hence enough to show

$$\mathbf{E}^{\uparrow}_x\left[\int_0^{\infty} \Phi\left(\lambda e^{-\overline{K}_s}\right) ds\right] < \infty,$$

THANK YOU FOR YOUR ATTENTION