STOCHASTIC FIXED POINTS on D

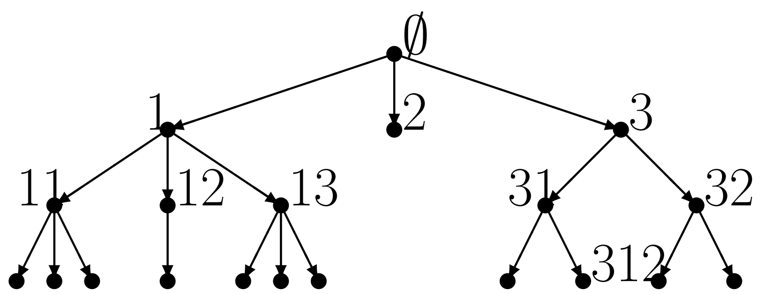
Uwe Roesler

- **Weighted Branching Process**
- **Stochastic Fixed Points on $\mathbb{R}$**
- **Stochastic Fixed Points on D**
- **chasing all**
- **Example Quicksort process**
  If necessary
- **Quicksort Process**
- **Discrete Partial Quicksort**
- **Convergence results**

WEIGHTED BRANCHING PROCESS

Uwe Rösler

Tree V with Ulam-Harris notation



Random weights $L_e : \Omega \to G$ on edges $e = (v, vi)$,

$(G, *)$ a semi group (with grave), often $G \subset H^H$ with composition

Random weights $L_e : \Omega \to G$ on pathes $e = (v, vw)$

Weighted Branching Process iff $(L_{v,vi})_{i \in \mathbb{N}},\ v \in V$ iid

Ex: **BGW, BRW**

Ex: **Branching Processes** or **Markovian BP**

Ex: **BP with continuous time**, $L_{v,vi}$ as life time,

Ex: **Splitting processes**,

– interval splitting

– Kolmogorov rock crushing

– application in biology (genealogical tree)

– algorithms (divide and conquer algorithms, Quicksort)

Ex: **Kingman process** as time transformed splitting process

WBP, dynamic is splitting a subset $A \subset \mathbb{N}$ into two

– choose $p \in [0,1]$ uniform distribution

– take iid Bernoulli $B_i, i \in \mathbb{N}$ to parameter p

– $A_1 := \{a \in A \mid B_a = 1\} \qquad A_2 = A \backslash A_1$

– start with $\mathbb{N}$ on root

Symmetry of edges and vertices, $V \ni v \mapsto (L_{v,vi})_i$

Skip root $\emptyset$, e.g. $v = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^*$, $L_v = L_{\emptyset,\emptyset v}$

STOCHASTIC FIXED POINTS

Stochastic fixed point equation (**SFPE**)

$$X \stackrel{\mathcal{D}}{=} f(X_1, X_2, X_3, \ldots)$$

where X iid rvs and random variable f independent of X-rvs.

The distribution of X is a solution (**SFP**).

Why interested in **distribution**? Obviously exists versions of solution satisfying

$$X = f(X_1, X_2, X_3, \ldots)$$

Easier and.... For simplicity on $I\!R$ and f linear or affin

$$X \stackrel{\mathcal{D}}{=} \sum_{i \in I\!N} A_i X_i \qquad X \stackrel{\mathcal{D}}{=} \sum_{i \in I\!N} A_i X_i + C$$

homogeneous sum-type and inhomogeneous sum-type. C stands for cost or toll.

Iterate inhomogeneous equation with equality

$$\begin{aligned} X &= C + \sum_{i \in \mathbb{N}} A_i C_i + \sum_{i,j \in \mathbb{N}} A_i A_{i,j} X_{i,j} = \dots \\ &= R_n + \sum_{|v|=n} L_v X_v \\ R_n &= \sum_{|v|<n} L_v C_v \end{aligned}$$

Choose $((L_{v,vi})_{i \in \mathbb{N}}, C_v),\ v \in V$ as iid. Show $R_n \to_n R$ a.e. (contraction method or martingale) and $\sum_{|v|=n} L_v X_v \to 0$ disappears in distribution. Possible. Solution is R. **Endogenous** solution. **Forward** solution. Boundary at root. Probability.

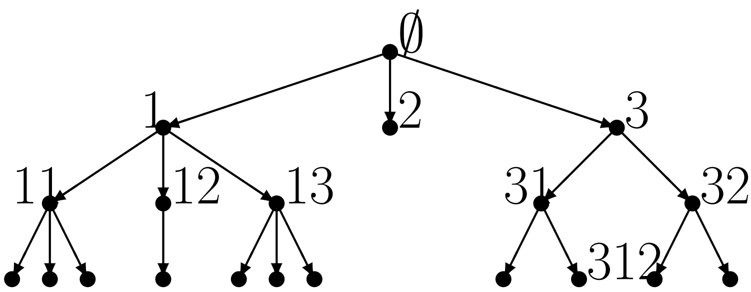
Iterate homogeneous equation with equality

$$X = \sum_{i,j \in \mathbb{N}} A_i A_{i,j} X_{i,j} = \ldots = \sum_{|v|=n} L_v X_v$$

For every n exists versions of $X_v,\ |v| = n$ satisfying the above with equality. But in the limit???? If we consider only distributions then by Kolmogorovs projective limit exists a projective distribution. That does the job.

Backward solution. Boundary at infinity. Measure theory.

non endogenous solutions, influence of boundary



MOTIVATION for SFP

Ex: Divide and conquer algorithm provides

$$X_{n+1} \stackrel{\mathcal{D}}{=} \sum_i A_{n,i} X_{n,i} + C_n$$

assume ...

$$X \stackrel{\mathcal{D}}{=} \sum_i A_i X_i + C$$

Solve. Limiting object.

Ex: **α-stable-distributions** $\alpha \in (0, 2]$ (=Limit of sums of iid)

X α-stable distribution iff $X \stackrel{\mathcal{D}}{=} aX_1 + bX_2 + c$ for all $a, b \in \mathbb{R}$ satisfying $|a|^\alpha + |b|^\alpha = 1$ and exists c.

Replace for all by some specific a, b, c. Is solution $\alpha-$stable? Unique solution? Alsmeyer-Roesler '05

Ex: **Quicksort** Result on asymptotic running time. Recursion for running time. Limit is SFPE.

$$X \stackrel{\mathcal{D}}{=} UX_1 + (1 - U)X_2 + C(U)$$

$$C(x) = 2x \ln x + 2(1 - x) \ln(1 - x) + 1$$

L^2-Solution Roesler 91,92.

ALL SFPs on the REALS

Motivation for studying SFPs in its own arose by Quicksort. With contraction method invented there, 30-40 different divide and conquer algorithms could be treated, Neininger, Rüschendorf, ... '01, '02, '04, '06 ... Metrics Wasserstein and Zolotarev

Rueschendorf: General solution of inhomogeneous is sum of general solution of homogeneous and some special solution of inhomogeneous.

Fill and Janson '00 found all solutions for Quicksort SFE, symmetric 1-stable.

Jagers and Roesler '04 Supremums and infimums type equations

SFPE on $I\!R_+$ and finitely many weights, Liggett, Liu

on $I\!R$ Caliebe '03, finite variance and finite first moment, (via triangle scheme)

Spitzmann: Partial results on $I\!R$ and connection with inhomogeneous solutions

General solution on $I\!R$ for finite number of weights, Alsmeier-Biggins-Meiners '12 Iff conditions for all solutions. These are mixtures of α-stable distributions. α determined by $m(\alpha) = 1$ where $\beta \to m(\beta) = E(\sum_i |L_i|^\alpha)$

Solutions are

$$R + W^{1/\alpha} Y$$

$$W \stackrel{\mathcal{D}}{=} \sum_i |L_i|^\alpha W_i$$

Solutions in $I\!R^d$ Mentemeier '12

CADLAG-PROCESSES as SFPs

$X = (X(t))_{t\in[0,1]}$ with cadlag pathes

$$X \stackrel{\mathcal{D}}{=} f(X_1, X_2, X_3, \ldots)$$

$f, X_i,\ i \in \mathbb{N}$ independent, every X_i distributed as X

Restrict us to affin f

$$X \stackrel{\mathcal{D}}{=} \sum_j T_j * X_j + C$$

where T_j is random operator on D, space and time transformation

Ex: **Brownian motion**

$$X \stackrel{\mathcal{D}}{=} \sqrt{U}X_1 + \sqrt{1-U}X_2$$

U another independent rv, values in $[0,1]$ **or**

$$X \stackrel{\mathcal{D}}{=} (UX_1(\frac{t}{U} \wedge 1) + (1-U)X_2(\frac{t-U}{1-U} \vee 0))_t$$

Ex: **Cauchy**

$$X \stackrel{\mathcal{D}}{=} UX_1 + (1-U)X_2$$

solved by symmetric Cauchy(b) distribution, density $x \mapsto \frac{b}{\pi(b^2+x^2)}$ and by constants. **Or**

$$X \stackrel{\mathcal{D}}{=} (U^2X_1(\frac{t}{U} \wedge 1) + (1-U)^2X_2(\frac{t-U}{1-U} \vee 0))_t$$

Ex: **Levy processes** Analogous space-time transformation

Ex: **Find** Analysis of algorithms,
Gruebel-Roesler '96

$$X \stackrel{\mathcal{D}}{=} (\mathbb{1}_{t<U)} U X_1(\frac{t}{U}) + \mathbb{1}_{t \geq 1}(1-U) X_2(\frac{t-U}{1-U}) + 1$$

U uniformly distributed on $[0,1]$

Neininger-Sulzbach '12, general functional contraction method with Zolotarev metric on D

QUICKSORT on the FLY

Conrado Martinéz: **Partial Quicksort**
Input: sequence of length n
Output: l smallest in order
Procedure: Recall Quicksort always for left most list with 2 or more elements
Publish first smallest then second smallest and so on

Observation: Algorithms does only necessary comparisons
$Y(n,l)$ number of comparisons for input $U_{|n}$
$X(n, \frac{l}{n}) = \frac{Y(n,l) - EY(n,l)}{n}$
Theo Martinéz '04
Explicit formula for $E(Y(n,l))$

Theo Roesler '13
$X(n,\ldots)$ converges in Skorodhod metric almost surely to Quicksort process.

SEARCH for EXAMPLES

Quicksort, limit SFP, **All** solutions Fill-Janson '00

Quicksort process on D by contraction method

$$X \stackrel{\mathcal{D}}{=} (\mathbb{1}_{t<U} U X_1(\frac{t}{U} \wedge 1) + (1-U) X_2(\frac{t-U}{1-U}) + C(U,t))_t$$

U uniform independent Discrete to continuous process, Knof '06 for finite dimensional distribution, Ragab-Roesler '11, convergence in distribution, Roesler '13 point wise in Skorodhod

Again endogenous and non endogenous solutions

Better work with caglad functions (caglad =left continuous with right limits)

Here functions on $[0,1]$ and $X(0) = 0$

$$X \stackrel{\mathcal{D}}{=} (U X_1(\frac{t}{U} \wedge 1) + (1-U) X_2(\frac{t-U}{1-U} \vee 0) + C(U,t))_t$$

ALL CADLAG-PROCESSES as SFPs

Class of processes

$$X \stackrel{\mathcal{D}}{=} \sum_j A_j \cdot X_j \circ \varphi + C$$

A_j, C rvs D-valued, φ random time change

How many solutions has the Quicksort process SFPE?

For Quicksort: sum of ordinary QD and symmetric Cauchy distribution, Fill-Janson '00

Theo Roesler '18

For Quicksort process: sum of QP and symmetric Cauchy process

$$R + a\mathrm{Id} + bY$$

with $a, b \in \mathbb{R},\ b \geq 0,\ R, Y$ independent, R solves the inhomogeneous SFPE and Y is a standard Cauchy process.

The standard Cauchy process solves the homogeneous SFPE

$$X \stackrel{\mathcal{D}}{=} (UX_1(\frac{t}{U} \wedge 1) + (1-U)X_2(\frac{t-U}{1-U} \vee 0))_t$$

Idea of proof: Take view of equality of random variables. and iterate via WBP

$$Y_v = (\mathbb{1}_{U_v>t} U Y_{v1}(1 \wedge \frac{t}{U_v}) + \mathbb{1}_{U_v \leq t}(1-U_v)Y_{v2}(\frac{t-U_v}{1-U_v})))_t$$

Consider equations at time $t = 1$

$$Y_v(1) = U_v Y_{v1}(1) + (1 - U_v)Y_{v2}(1)$$

If we know all $Y_v(1)$ then process Y uniquely defined at the points $Y(t)$ where t are the times (places) of the pivots. These are dense in the limit and by continuity the path of Y is known.

DIRTY RECURSIONS

Recursion for Y

$Y(x,l) = Y(x^1, l \wedge |x^1|) + Y(x^2, 0 \vee (l - |x^1| - 1)) + |x| - 1$ and then take $x = U_{|n}$. Recursion for $X(n)$

Real strength of contraction method shows up for 'dirty' recursions

$$X(n) \stackrel{\mathcal{D}}{=} \sum_i L_i(I(n)) X_i(I(n)) + C(I(n))$$

$(L_i(\cdot))_i, C(\cdot), I(n)), X(j),\ j < n$ independent, $I(n) < n$

Assume: $I(n) \to_n \infty,\ L_i(n) \to_n L_i,\ C(n) \to_n C$
Hope: $X(n) \to X$ and

$$X \stackrel{\mathcal{D}}{=} \sum_i L_i X_i + C$$

With 'nice' metric d on distributions
$d(X(n), X)$ function of $d(L_i(I(n)), L_i), d(X_i(I(n)), X_i), d(C(I_n$
shows convergence to 0.

CONVERGENCE of finite dimensional DISTRIBUTIONS

Theorem Martinez-Roesler
The one-dimensional distributions of $X(n)$ converge.

Theorem Ragab-Roesler
All finite dimensional distributions of $X(n)$ converge to the ones of the Quicksort process.

As consequence exists versions of $X(n)$ converging in Skorodhod metric to version of QP
Find algorithmic versions.
Theorem Roesler Let $U_i,\ i \in I\!N$, be independent uniformly distributed. Then $X(n, \cdot)$ converges almost surely to a version of the Quicksort process X in Skorodhod metric on D.

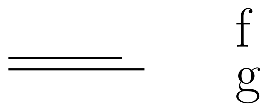
Deterministic algorithm, but random input.

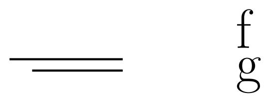
SKORODHOD SPACE D

D equipped with Skorodhod metric d

$$d(f,g) = \inf\{\epsilon > 0 \mid \exists \lambda \in \Lambda : \|f - g \circ \lambda\|_\infty < \epsilon,\ \|\lambda - \mathrm{id}\|_\infty < \epsilon\}$$

where Λ is the set of all bijective increasing functions $\lambda : [0,1] \to [0,1]$.





Big distance in supremum metric, small in Skorodhod metric

Theo: Alsmeyer-Biggins-Meiners

$N = \Sigma_i \, 1\!\!1_{L_i \neq 0} < \infty$ a.e., distribution on $I\!\!R$

– $(0, \infty)$ is smallest closed multiplicative group generated by strictly positive factors

– $m(0) > 1$

– $\exists \alpha \in (0, 2] : m(\alpha) = 1$

– $\forall 0 < \beta < \alpha : 1 < m(\beta)$

Then $\psi(t) = \Pi_{|v|=n} \psi_v(L_v t)$ L-a.e. for all t by martingale argument for Fourier transform. For given L

– infinitely divisible distribution,

– parameters in Levy representation satisfy fixed point equation,

– then stable.

$$\ln \psi(t) = \begin{cases} -cW|t|^\alpha (1 - i\beta \frac{t}{|t|} \tan(\frac{\pi\alpha}{2})) & \text{if} \quad \alpha \notin \{1, 2\} \\ i\gamma W t - cW|t| & \text{if} \quad \alpha = 1 \\ -\sigma^2 W t^2 & \text{if} \quad \alpha = 2 \end{cases}$$

Remark: $N = \infty$?

Mixtures of stable distributions

INHOMOGENOUS SFE

Best result on the reals

Theo Meiners ’10

Assumptions as before and on C

Set of **all** solution are distributions of

$$W^* + W^{1/\alpha} Y$$

where

– $(W^*, W), Y$ are independent,

– W^* is one solution of inhomogeneous SFE,

– $W \geq 0$ solves $W \stackrel{\mathcal{D}}{=} \Sigma_j \, |L_j|^\alpha W_j$

– Y has stable distribution to parameters

$$(\gamma, c, \beta) \in \begin{cases} \{0\} \times [-1,1] \times [0,\infty) & \text{if} \quad \alpha \notin \{1,2\} \\ \mathbb{R} \times \{0\} \times [0,\infty) & \text{if} \quad \alpha = 1 \\ \{0\} \times \{0\} \times [0,\infty) & \text{if} \quad \alpha = 2 \end{cases}$$

SFE of SUPREMUM TYPE

On positive reals

$$X \stackrel{\mathcal{D}}{=} \sup_j L_j X_j + C$$

Jagers-Roesler, more than those expected by SFE of sum type

Rueschendorf, Alsmeyer

'complete' results