Statistical inference for non-explosive branching processes based on partial observations

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Outline

Introduction

Process $W_n, n \geq 0, W_0 = 1$, defined by two families of independent, nonnegative integer valued random variables $\{X_{ni}, (n,i) \in \mathcal{N}_0 \times \mathcal{N}\}, \mathcal{N} = \{1, 2 ...\}, \mathcal{N}_0 = \mathcal{N} \cup \{0\}$ and $\{\nu_k, k \geq 1\}$ recursively as

$$W_{n+1} = \sum_{i=1}^{W_n} X_{ni} + \nu_{n+1}, \quad n \geq 0. \tag{1}$$

Assume: X_{ni} have a common distribution for all n and i;
Families $\{X_{ni}\}$ and $\{\nu_n\}$ are independent;

- $\{\nu_k, k \geq 1\}$ are independent but not necessarily identically distributed random variables.

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Partially observed process

Let
$\{\xi_{ni}, (n,i) \in \mathcal{N}_0 \times \mathcal{N}\}$ family of i.i.d. Bernoulli random variables with a probability of success θ
$\{X_{ni}^{(j)}, (n,i) \in \mathcal{N}_0 \times \mathcal{N}\}, j = 1, 2$ are independent families of i.i.d. random variables taking nonnegative integer values and these families may follow different probability distributions for $j = 1, 2$. Assume also that families $\{\xi_{ni}, (n,i) \in \mathcal{N}_0 \times \mathcal{N}\}$ and $\{X_{ni}^{(j)}, (n,i) \in \mathcal{N}_0 \times \mathcal{N}\}$ are independent for all values of n, i and j. We take

$$X_{ni} = X_{ni}^{(1)}(1 - \xi_{ni}) + X_{ni}^{(2)}\xi_{ni}. \quad (2)$$

- In epidemic modeling: "Quarantine assumption".

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We take

$$X_{ni} = X_{ni}^{(1)}(1 - \xi_{ni}) + X_{ni}^{(2)}\xi_{ni}. \tag{2}$$

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Partially observed process

Obtain new branching process with immigration $Z_0 = 1$,

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{ni}^{(1)}(1-\xi_{ni}) + \sum_{i=1}^{Z_n} X_{ni}^{(2)}\xi_{ni} + \nu_{n+1} \quad n \geq 0.$$

The partially observed branching process with immigration is now defined as

$$Y_{n+1} = \sum_{i=1}^{Z_n} \xi_{ni}, \; n \geq 0.$$

- "Binomial thinning".
- Inspection changes the offspring distribution of an individual.

Partially observed process

References:

• Meester R., De Koning J., De Jong M., S., Diekmann O. (2002) Biometrics, **58,** 178-184.

• Meester R., Trapman P. (2006) Appl. Probab., **38,** 1098-1115.

• Panaretos V., M.(2007) Partially observed branching processes for stochastic epidemics, J. Math. Biol., **54,** 645-668.

• Kvitkovičov*á* A., Panaretos V., M.(2011) Adv. Appl. Probab. **43,** 1166-1190.

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Estimators

What is the sample? It is $\{(W_i, \nu_i), i = 1, 2, ...n\}$, if the process fully observed. The estimator for m given by (Nanthi(1979), Venkatarman (1982) and more..)

$$\hat{m}_n = \frac{\sum_{i=1}^{n}(W_i - \nu_i)}{\sum_{i=1}^{n} W_{i-1}}. \tag{3}$$

If the process is partially observed then the sample:

$$\{(Y_i, \eta_i), i = 1, 2, ...n\},$$

where η_i are the number of observed immigrants:

$$\eta_{n+1} = \sum_{j=1}^{\nu_n} \xi_{nj}, \; n \geq 1,$$

Estimators

Our estimators for m are given by

$$\hat{a}_n = \frac{\sum_{i=2}^{n+1}(Y_i - \eta_i)}{\sum_{i=1}^{n} Y_i} \tag{4}$$

based on the partial observations of the reproduction and the immigration processes.

We also consider modified estimators for m defined as follows

$$\hat{b}_n = \frac{\sum_{i=1}^{n_o}(Y_{2i+1} - \eta_{2i+1})}{\sum_{i=1}^{n_o} Y_{2i}}, \ \hat{c}_n = \frac{\sum_{i=1}^{n_e}(Y_{2i} - \eta_{2i})}{\sum_{i=1}^{n_e} Y_{2i-1}}, \tag{5}$$

where $n_o = [(n-1)/2]$ and $n_e = [n/2]$.

- Why modified estimators? We talk about it little later.

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In fully observed processes consistency and normality of the estimators depend on asymptotic properties of the process. So we look the problem from a little different point. Namely, we try to answer the following question.

- Which asymptotic properties of the process are essential for estimators to be consistent and asymptotically normal?
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Consistency

First we talk about consistency. We assume:

A1. $m \in (0, \infty)$ *and* $\sigma^2 < \infty$.

A2. The sum $\sum_{i=1}^{n} \nu_i \xrightarrow{d} \infty$ *as* $n \to \infty$ *and almost surely*

$$\limsup_{n\to\infty} \frac{\nu_n}{\sum_{i=1}^{n-1} \nu_i} < \infty.$$

- About conditions A1 and A2.

Condition A1 is natural.

Condition A2. First part: The total number of immigrating individuals increases. Second part: excludes the situation when the number of individuals immigrating to a single generation predominates the total number of immigrants to all other generations.

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The following theorem holds.

Theorem 1. *If conditions A1 and A2 are satisfied, then* $\hat{a}_n$ *defined in* (4) *is a strongly consistent estimator for* m*.*

We now consider the situations when conditions of Theorem 1 are fulfilled.

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Consistency

Examples of application.

1.Stationary immigration.

Strong Law of Large numbers: $\sum_{i=1}^{n} \nu_i \xrightarrow{d} \infty$ as $n \to \infty$ whenever $\lambda_1 \in (0, \infty)$, where $\lambda_i = E\nu_i$.

Also: This fact and again SLLN: $\nu_n/n \to 0$ a.s. as $n \to \infty$. Thus condition A2 is also satisfied.

Corollary 1. *If the immigration is stationary, $m \in (0, \infty)$, $\sigma^2 < \infty$ and $\lambda_1 \in (0, \infty)$, then $\hat{a}_n$ is a strongly consistent estimator for m.*

Consistency

2. Non-Stationary immigration.

- Let R_a the set of regularly varying functions with exponent a.
- Denote: λ_i and γ_i^2 are mean and variance of ν_i respectively.
- Assume that: $(\lambda_i)_{i=1}^{\infty} \in R_\lambda$ and $(\gamma_i^2)_{i=1}^{\infty} \in R_\gamma$ for some $\lambda, \gamma \geq 0$.

Notation: $\Lambda_n = \sum_{i=1}^{n} \lambda_i$, $\Gamma_n = \sum_{i=1}^{n} \gamma_i^2$.

Theorem 2. *If $m \in (0, \infty)$, $\sigma^2 < \infty$, $\Lambda_n \to \infty$, $\Gamma_n = o(\Lambda_n^2)$ as $n \to \infty$ and series $\sum_{i=1}^{\infty} \gamma_i^2 \Lambda_{i-1}^{-2}$ is convergent, then $\hat{a}_n$ is a strongly consistent estimator for m.*

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Consistency

Remarks. Conditions of Theorem 2 are satisfied if
a) $\gamma_i^2, i \geq 1$ are uniformly bounded and $\Lambda_n \to \infty$.
b) Another situation when the conditions hold is $\gamma_n^2 = O(\lambda_n)$.
c) Let now $\gamma_n^2 = \lambda_n \Lambda_{n-1}^{\delta}$ for some $0 \leq \delta < 1$. Since $2 - \delta > 1$, using properties of regularly varying functions one can show that $\Gamma_n = o(\Lambda_n^2)$. On the other hand series $\sum_{i=1}^{\infty} \gamma_i^2 \Lambda_{i-1}^{-2} = \sum_{i=1}^{\infty} \lambda_i \Lambda_{i-1}^{\delta-2}$ is convergent due to Dini's theorem (Knop (1956), p. 125).

Consistency

Example 1. Let $\nu_k, k \geq 1$ be Poisson with mean λ_k. In this case trivially $\Gamma_n = o(\Lambda_n^2)$ as $n \to \infty$. It again follows from Dini's theorem that series $\sum_{i=1}^{\infty} \lambda_i \Lambda_{i-1}^{-2}$ is convergent and we obtain the following result from Theorem 4.

Corollary 2. *If $m \in (0, \infty)$, $\sigma^2 < \infty$, and $\nu_k, k \geq 1$ is Poisson with mean λ_k such that $\Lambda_n \to \infty$ as $n \to \infty$, then $\hat{a}_n$ is a strongly consistent estimator for m.*

Asymptotic Normality

- Key properties in fully observed process: $W_n - mW_{n-1} - \lambda_n$ is a martingale difference or $\{W_n, n \geq 0\}$ is a Markov chain.
- For the partially observed process Y_n these properties do not hold. Therefore it was not possible to get asymptotic normality of the original estimator $\hat{a}_n$. We modify it using "skipping one index" method and consider estimators $\hat{b}_n$ and $\hat{c}_n$.
- The "skipping" idea belongs Meester R., Trapman P. (2006). However their result: Normed difference of the estimator and parameter is asymptotically sum of three normal random variables (not independent).

Asymptotic Normality

- In Kvitkovičov*á* A., Panaretos V., M.(2011): In the proof of normality they use Scott's Central Limit theorem on convergence of a random sequence (with martingale difference components) to a Gaussian measure.
- Scott D. J. (1978). A central limit theorem for martingales and an application to branching processes. Stoch. Process. Appl. **6,** 241-252.

Asymptotic Normality

Our representation:

$$(\hat{b}_n - m)\sum_{i=1}^{n_o} Y_{2i} = \sum_{k=1}^{n_o}\sum_{i=1}^{Z_{2k-1}} \omega_{ki}, \tag{6}$$

where $n_o = [(n-1)/2]$ and $\omega_{ki} = \sum_{j=1}^{X_{2k-1i}} \rho_{2k,ij} - m\xi_{2k-1i},\ i \in \mathcal{N}$. Here $\rho_{kij}, (k,i,j) \in \mathcal{N}^3$ are i.i.d. Bernoulli random variables with the probability of success θ.

- If not modified estimator, $(\hat{a}_n - m)\sum_{i=1}^{n} Y_i$ and these random variables are not independent.

Asymptotic Normality

A3. There exists a sequence of positive integers $\{c_k, k \geq 1\}$ *and constant* $C \in (0, \infty)$ *such that* $c_n \to 0$ *as* $n \to \infty$ *and*

$$c_n \sum_{i=1}^{n} W_{2i-1} \xrightarrow{P} C.$$

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Theorem 3.
a) If conditions A1 and A3 are satisfied, then

$$(\sum_{k=1}^{n_o} Y_{2k})^{1/2}(\hat{b}_n - m) \xrightarrow{d} N(0, b^2)$$

as $n \to \infty$, *where* $b^2 = m(1-\theta) + m^2(1+\theta) + \sigma^2\theta - 2m\theta EX_{ki}^{(2)}$.
b) If conditions A1 and A4 are satisfied, then as $n \to \infty$

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There are examples of processes for which the conditions of Theorem 3 are satisfied.

a) Processes with the stationary immigration in subcritical case.

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A. Pakes (1971): if $m < 1$ and $\lambda_1 \in (0, \infty)$, then $n^{-1} \sum_{i=0}^{n} W_i \to \lambda_1/(1-m)$ a.s. as $n \to \infty$. So one can prove that condition A3 is satisfied. Hence we have the following result from Theorems 2 and 3.

Theorem 4. *If the immigration is stationary, $m < 1$, $\sigma^2 < \infty$ and $\lambda_1 \in (0, \infty)$, then $\hat{a}_n$ is a strongly consistent estimator for m and as $n \to \infty$*

$$\sqrt{n}(\hat{b}_n - m) \xrightarrow{d} N(0, \frac{2b^2(1-m)}{\lambda_1 \theta}).$$

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Theorem 5. *If $m < 1$, $\sigma^2 < \infty$, $\lambda_n \to \infty$ and $\gamma_n^2 = o(\lambda_n^2)$ as $n \to \infty$, then as $n \to \infty$*

$$\sqrt{nA(n)}(\hat{b}_n - m) \xrightarrow{d} N(0, \frac{2b^2(1+\lambda)}{\theta}).$$

Theorem 6. *If $m = 1$, $\sigma^2 < \infty$, $\lambda_n \to \infty$ and $\gamma_n^2 = o(n\lambda_n^2)$ as $n \to \infty$, then as $n \to \infty$*

$$\sqrt{nA(n)}(\hat{b}_n - m) \xrightarrow{d} N(0, \frac{2b^2(2+\lambda)}{\theta}).$$

- There is **Theorem 7** related to supercritical case.

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Large number of ancestors

- No immigration: $\nu_n = 0, \ n \geq 0.$

- Large number of initial individuals: $W_0 = n.$

- Yanev, N.M., 1976. On the statistics of branching processes. Theory Probab. Appl., 20, 612-622.

To estimate the offspring mean based on partial observations, we denote the partially observed branching process started by n initial ancestors by $\mu_n(t), \ t \geq 0, \ n \geq 1$ and define estimator

$$\hat{a}_n(t) = \frac{\sum_{k=2}^{t+1} \mu_n(k)}{\sum_{k=2}^{t+1} \mu_n(k-1)}. \tag{7}$$

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We provide only the result on the consistency of this estimator.

Theorem 8. *a) If $m < 1$, then $\hat{a}_n(t)$ is a consistent estimator for m i.e. $\hat{a}_n(t) \to m$ in probability as $n, t \to \infty$.*
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