The continuous-state nonlinear branching process

Pei-Sen Li

A talk based on L, Yang and Zhou (2018) and L and Zhou (2018)

(2018/4/12)

Why do we study nonlinear branching process?

"Population growth of harbor seals in Washington State"

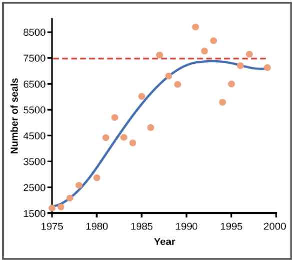


Image credit: "Environmental limits to population growth: Figure 2"

Obviously, interactions between different individuals exist in the nature.

The evolution is also influenced by some random factors.

Motivation: We want to model the interactions between individuals and stochastic perturbations by environment.

Controlled branching process

In order to describe competition and interaction between individuals, the controlled Galton-Watson branching process is defined by

$$X_n = \sum_{i=1}^{r(X_{n-1})} \xi_{n,i}, \qquad n \geq 1, \tag{1}$$

where the function $r(x) \geq 0$ controls the reproduction and $\xi_{n,i}$ are i.i.d. random variable.

A slight generalization of the above model is defined by:

$$X_n = \sum_{i=1}^{r(X_{n-1})} \xi_{n,i} + g(X_{n-1}), \tag{2}$$

where the function $g(x) \geq -x$ represents the influence of competition, interaction, immigration and so on.

Our goal is to construct and to study the continuous-state and continuous-time version of (2) by using SDE with jumps.

Recall that a generalization of the controlled Galton-Watson process is defined by:

$$X_n = g(X_{n-1}) + \sum_{i=1}^{r(X_{n-1})} \xi_{n,i}. \tag{3}$$

Suppose that $\mu := \mathbb{E}(\xi_{1,1}) < \infty$. Then $[f(x) := g(x) + \mu\gamma(x) - x]$:

$$X_n - X_{n-1} = g(X_{n-1}) + \mu r(X_{n-1}) - X_{n-1} + \sum_{i=1}^{r(X_{n-1})} (\xi_{n,i} - \mu),$$

$$X_n = X_0 + \sum_{k=1}^{n} f(X_{k-1}) + \sum_{k=1}^{n} \sum_{i=1}^{r(X_{k-1})} (\xi_{k,i} - \mu).$$

A typical continuous-state nonlinear branching process is the solution of:

$$x(t) = x(0) + \int_0^t f(x(s-))ds + \int_0^t \int_0^{r(x(s-))} \int_0^\infty \xi \tilde{N}(ds, du, d\xi), \tag{4}$$

where $\tilde{N}(ds, du, d\xi)$ is a compensated Poisson random measure on $(0, \infty)^3$.

Continuous-state nonlinear branching process

A general continuous-state nonlinear branching process is constructed by the SDE:

$$\begin{aligned} X_t = X_0 + \int_0^t \gamma_0(X_s)ds + \int_0^t \int_0^{\gamma_1(X_s)} W(ds, du) \\ + \int_0^t \int_0^{\gamma_2(X_{s-})} \int_0^\infty z\tilde{N}(ds, du, dz). \end{aligned} \quad (5)$$

where $W(ds, du)$ is a Gaussian white noise based on $dsdu$ and $\tilde{N}(ds, du, dz) = N(ds, du, dz) - dsdu\pi(dz)$ is a compensated Poisson random measure $\left(\int_0^\infty z \wedge z^2\pi(dz) < \infty\right)$. The red term is a continuous parallel of the blue term.

Here γ_0 describes the interaction and $\gamma_1 \geq 0, \gamma_2 \geq 0$ describe the perturbations.

By saying $\{X_t : t \geq 0\}$ is a solution to (5), we mean it satisfies (5) before hitting 0 or ∞ and it is trapped by those states.

Proposition (L, Yang and Zhou 2018)

For locally Lipschitz functions γ_1, $\gamma_2 \geq 0$ and γ_0 on $(0, \infty)$ with $\gamma_i(0) = 0$ $(i = 0, 1, 2)$, there exists a pathwise unique positive solution to (5).

Example: continuous-state branching process

Let $\gamma_i(x) = b_i x$, $b_1 > 0, b_2 = 1$. Then the solution to

$$X_t = X_0 + b_0 \int_0^t X_s ds + b_1 \int_0^t \int_0^{X_s} W(ds, du) + \int_0^t \int_0^{X_s} \int_0^\infty z\tilde{N}(ds, du, dz),$$

is a continuous-state branching process. This process was studied by Lamperti (1967) and many others.

Stochastic equations of this type have been studied by Berton and Le Gall (2006) and Dawson and Li (2006, 2012).

Only in this case, the solution to (5) satisfies branching property, which means that different individual act independently with each other.

Example: Feller branching diffusion with logistic growth

Lambert (2005) studied logistic branching process.

A special case of his model is the Feller branching diffusion with logistic growth defined by:

$$X_t = X_0 + \int_0^t (rX_s - cX_s^2)ds + \sigma \int_0^t \int_0^{X_s} W(ds, du)$$

or, equivalently,

$$X_t = X_0 + \int_0^t (rX_s - cX_s^2)ds + \sigma \int_0^t \sqrt{X_s} dB_s.$$

where $\sigma \geq 0$, $c \geq 0$ and b are constants.

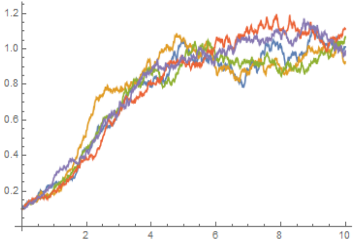
1. When $c = 0$, it reduces to a Feller branching diffusion; Feller (1951).
2. When $\sigma = 0$, it reduces to deterministic logistic growth model.

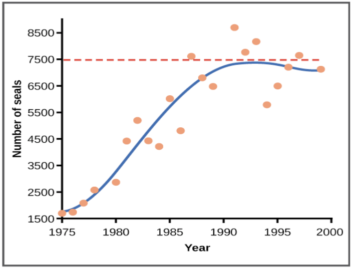
Example: Feller branching diffusion with logistic growth

Five sample paths of the Feller branching diffusion with logistic growth (by Mathematica 11):

$$X_t = 0.1 + \int_0^t (X_s - X_s^2) ds + 0.05 \int_0^t \sqrt{X_s} dB_s.$$

The sample paths are very similar to the curve of the number of seals.





Rich structures and problems to study

Continuous-state nonlinear branching processes involve rich mathematical structures and there are quite a number of problems to study.

For $x \geq 0$ and $y \geq 0$ let $\mathbb{P}_x(\cdot) = \mathbb{P}(|X_0 = x)$ and define the stopping times

$$\tau_y^- = \inf\{t > 0 : X_t \leq y\}, \quad \tau_y^+ = \inf\{t > 0 : X_t \geq y\}$$

and $\tau_\infty^+ = \lim_{y\to\infty} \tau_y^+$.

1. Can the process hit 0 in finite time with a positive probability? i.e. $\mathbb{P}_x\{\tau_0^- < \infty\} = 0$ or > 0?
2. Can the process hit ∞ in finite time with a positive probability? i.e. $\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$ or > 0?
3. Can the process come down from infinity? i.e. $\lim_{y\to\infty} \lim_{x\to\infty} \mathbb{P}_x\{\tau_y^- < t\} = 1$ for all $t > 0$?

(If it comes down from infinity, there exists $(X_t)_{t>0}$ so that $X_{0+} = \infty$.)

Martingale approach

The study of the above 3 questions is based on the local martingale $(a > 0)$:

$$X_t^{1-a} \exp\Big\{ \int_0^t G_a(X_s) ds \Big\}, \qquad t \geq 0,$$

where

$$\begin{aligned} G_a(u) &= \frac{(a-1)}{u}\gamma_0(u) - \frac{a(a-1)}{2u^2}\gamma_1(u) \\ &\quad - \gamma_2(u) \int_0^\infty \Big[\Big(1 + \frac{z}{u}\Big)^{1-a} - 1 - \frac{(1-a)z}{u}\Big]\pi(dz). \end{aligned}$$

Main results

Theorem 1 (L, Yang and Zhou 2018)

(i) If there exist $a > 1$ and $r < 1$ so that $G_a(u) \geq -(\ln u^{-1})^r$ for all small enough $u > 0$, then $\mathbb{P}_x\{\tau_0^- < \infty\} = 0$ for all small enough $x > 0$.

(ii) If there exist $0 < a < 1$ and $r > 1$ so that $G_a(u) \geq (\ln u^{-1})^r$ for all small enough $u > 0$, then $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ for all small enough $x > 0$.

Theorem 2 (L, Yang and Zhou 2018)

(i) If there exist constants $0 < a < 1$ and $r < 1$ so that $G_a(u) \geq -(\ln u)^r$ for all u large enough, then $\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$ for all $x > 0$.

(ii) If there exist $a > 1$ and $r > 1$ so that $G_a(u) \geq (\ln u)^r$ for all u large enough, then $\mathbb{P}_x\{\tau_\infty^+ < \infty\} > 0$ for all large x.

Main results

Theorem 3 (L, Yang and Zhou 2018)

(i) If there exit constants $a > 1$ and $r < 1$ such that

$$G_a(u) \geq -(\ln u)^r$$

for all u large, then process X doesn't come down from infinity.

(ii) If there exit constants $0 < a < 1, r > 1$ such that

$$G_a(u) \geq (\ln u)^r$$

for all u large enough, then process X comes down from infinity.

Discussions

Applying above theorems in the special case $\gamma_i(x) = b_i x^{r_i}$, $r_i > 0$ $(i = 0, 1, 2)$, $b_1, b_2 \geq 0$ and $\pi(dz) = z^{-1-\alpha}dz$ $(1 < \alpha < 2)$, we can make the conclusions:

- Both large negative drift γ_0 and large perturbations γ_1, γ_2 near 0 can cause extinction.
- Explosion is only caused by large positive drift γ_0 near ∞. Large perturbations can prevent explosion.
- Both large negative drift γ_0 and large perturbations γ_1, γ_2 near ∞ can cause coming down from infinity.

Special case: necessary and sufficient conditions

Consider the special case with $r > 0$ and $\gamma_i(x) = b_i x^r$ $(i = 0, 1, 2)$. Let

$$\psi(\lambda) = -b_0\lambda + \frac{b_1^2}{2}\lambda^2 + b_2 + b_2 \int_{(0,\infty)} (\mathrm{e}^{-\lambda z} - 1 + \lambda z)\pi(dz).$$

Theorem 4 (L and Zhou 2018)

- (Extinction property) for all $x > 0$, we have $\mathbb{P}_x\{\tau_0^- < \infty\} > 0$ if and only if

$$\int^{\infty-} \frac{\lambda^{r-1}}{\psi(\lambda)} d\lambda < \infty.$$

- (Non-Explosion) for all $x > 0$, we have $\mathbb{P}_x\{\tau_\infty^+ < \infty\} = 0$ if and only if one of the following two conditions is satisfied: (i) $b_0 \leq 0$; (ii) $b_0 > 0$ and

$$\int_{0+} \frac{\lambda^{r-1}}{-\psi(\lambda)} d\lambda = \infty.$$

Special case: necessary and sufficient conditions

Theorem 5 (L 2016)

The process comes down from infinity if and only if $b_0 \leq 0$ and

$$\int_{0+} \frac{\lambda^{r-1}}{\psi(\lambda)} d\lambda < \infty.$$

Remark:

- For classical continuous-state branching process $(r = 1)$, the extinction condition was obtained by Grey (1974).
- For classical continuous-state branching process $(r = 1)$, the non-explosion condition was given by Kawazu and Watanabe (1971).
- The classical continuous-state branching process $(r = 1)$ does not come down from infinity.

Special case: a Lamperti transformation

Let $(Z_t)_{t\geq 0}$ be a spectrally positive Lévy process with Laplace exponent ψ.

Let $T_x^- := \inf\{t \geq 0 : Z_t \leq x\}$. Define

$$\eta(t) := \inf\left\{s \geq 0 : \int_0^{s\wedge T_0} Z_u^{-r} du > t\right\}, \quad t \geq 0.$$

Then by Lamperti's transformation

$$(X_t)_{t\geq 0} := (Z_{\eta(t)})_{t\geq 0}$$

is a continuous-state nonlinear branching process with $\gamma_i(x) = b_i x^r$ $(i = 0, 1, 2)$.

- If $T_0^- < \infty$, then $\tau_0^- = \int_0^{T_0^-} Z_s^{-r} ds$.
- If $T_0^- = \infty$, then $\tau_\infty^+ = \int_0^{T_0^-} Z_s^{-r} ds$.

We can convert the study of τ_0^- and τ_∞^+ to that of the integral $\int_0^{T_0^-} Z_s^{-r} ds$.

The Lévy process: finiteness of integral functional

Proposition (L and Zhou 2018)

Let f be a strictly positive function on $(0, \infty)$ satisfying $\sup_{x \geq \varepsilon} f(x) < \infty$ for any $\varepsilon > 0$. Then for any $x > 0$, the following statements are equivalent:

(i)

$$\mathbb{P}_x\Big(\int_0^{T_0^-} f(Z_s)ds < \infty | T_0^- < \infty\Big) > 0;$$

(ii)

$$\mathbb{E}_x\Big[\int_0^{T_0^-} f(Z_t)e^{-\lambda Z_t}dt | T_0^- < \infty\Big] < \infty \qquad \text{for some/each } \lambda > 0.$$

The Lévy process: finiteness of integral functional

Proposition (L and Zhou 2018)

Suppose that $\psi'(0) < 0$ and f is a non-increasing positive function on $(0, \infty)$. Then the following statements are equivalent:

(i)

$$\mathbb{P}_x\Big(\int_0^{T_0^-} f(Z_s)ds < \infty | T_0^- = \infty\Big) > 0 \qquad \text{for each } x > 0;$$

(ii)

$$\mathbb{E}_x\Big[\int_0^{T_y^-} f(Z_s)ds\Big] < \infty \qquad \text{for each } y \in (0, x) \text{ and } x > 0.$$

The Lévy process: finiteness of integral functional

From Theorem 2.7 (ii) in Kuznetsov et al. (2011) we obtain

Proposition (L and Zhou 2018)

Let $q = \sup\{\lambda > 0 : \psi(\lambda) = 0\}$. For any $x > y > 0$ and $r > 0$ we have:
(i)

$$\mathbb{E}_x\Big[\int_0^{T_y^-} Z_t^{-r}dt\Big] = \frac{1}{\Gamma(r)}\int_0^\infty \frac{e^{-(\lambda y+qx-qy)} - e^{-\lambda x}}{\psi(\lambda)}\lambda^{r-1}d\lambda.$$

(ii)

$$\mathbb{E}_x\Big[\int_0^{T_0^-} Z_t^{-r}e^{-\lambda Z_t}dt|T_0^- < \infty\Big] = \frac{1}{\Gamma(r)}\int_0^\infty \frac{1-e^{-(\lambda+s)x}}{\psi(s+\lambda+q)}s^{r-1}ds.$$

Finally, by giving the condition of the finiteness of the above integrations we can prove Theorem 4.

Ongoing work: Speed of coming down from infinity

For a continuous-state nonlinear branching process $(X_t)_{t>0}$ coming down from infinity ($X_{0+} = \infty$), we want to find a function v such that

$$\lim_{t \to 0} \frac{X_t}{v(t)} = 1.$$

The same question for some other processes has been studied for similar models by Berestycki et al. (2010) and Bansaye et al. (2015).

Some references

Dawson, D.A. and Li, Z.H. (2006): Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34**, 1103–1142.

Lambert, A. (2005): The branching process with logistic growth. *Ann. Appl. Probab.* **15**, 1506–1535.

Kuznetsov A., Kyprianou A.E., Rivero V. (2012): The Theory of Scale Functions for Spectrally Negative Lévy Processes. In: *Lévy Matters II. Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg.

Li, P.S. (2016): A continuous-state polynomial branching process. arXiv:1609.09593v2.

Li, P.S., Yang, X. and Zhou, X. (2018): A general continuous-state nonlinear branching process. arXiv:1708.01560.

Li, P.S. and Zhou, X. (2018): Integral functionals for spectrally positive Lévy processes. Preprint.

Pakes, A.G. (2007): Extinction and explosion of nonlinear Markov branching processes. *J. Austr. Math. Soc.* **82**, 403–428.

Thank you!