Extinction time of a CSBP with competition in a Lévy environment.

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Outline

CSBP with competition in a Lévy random environment.

Brownian case

Logistic competition.

Feller diffusion case.

CSBP with competition in a Lévy environment

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Consider a sequence of random environments $(E_n^N, n \geq 1)$ which are i.i.d. and define

$$Z_{n+1}^N = \sum_{i=1}^{F_N(Z_n^N)} L_{i,n}^N(Z_n^N, E_n^N) \quad \text{and} \quad S_{n+1}^N = S_n^N + E_n^N,$$

where for each (z, e), $(L_{i,n}^N(z,e), i \geq 1, n \geq 0)$ are i.i.d. with common distribution $L(z,e) \in \mathbb{N}$ a.s.

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What about the scaling limits of

$$\left(\frac{Z_{[v_N \cdot]}^N}{N}, S_{[v_N \cdot]}^N \right) ??$$

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More precisely we can introduce such family of processes as the unique strong solution of the following SDE (He, Li and Xu and Palau and P. (2018))

$$Z_t = Z_0 + b\int_0^t Z_s \mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t Z_{s-}\mathrm{d}S_s$$
$$+ \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where g is a continuous function on $[0,\infty)$ with $g(0) = 0$, $B^{(b)}$ is a standard Brownian motion $N^{(b)}$ is a Poisson random measure defined on $\mathbb{R}^3_+$ with intensity measure $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ such that

$$\int_{(0,\infty)} (1 \wedge z^2)\mu(\mathrm{d}z) < \infty,$$

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$$S_t^{(e)} = \gamma t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)^c} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)} (e^z - 1)\tilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z),$$

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with $\gamma \in \mathbb{R}, \sigma \geq 0$, $B^{(e)}$ is a standard Brownian motion and $N^{(e)}$ is a Poisson random measure taking values on $\mathbb{R}_+ \times \mathbb{R}$ and with intensity $\mathrm{d}s\pi(\mathrm{d}z)$ satisfying

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

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- Again the logistic case Clement Foucart will provide a complete understanding of the process, (2018).

Some motivation

We consider that the branching and competition mechanisms are as follows

$$g(x) = kx^2 \qquad \text{and} \qquad \psi(\lambda) = b\lambda \qquad \text{for } x, \lambda \geq 0,$$

where k is a positive constant.

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In particular, it can be rewritten as follows

$$Z_t = \frac{Z_0 e^{K_t}}{1 + kZ_0 \int_0^t e^{K_s}\mathrm{d}s}, \qquad t \geq 0,$$

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ii) *If the process K oscillates, then $\liminf_{t\to\infty} Z_t = 0$ a.s.*

iii) *If the process K drifts to ∞, then Z has a stationary distribution whose density can be written in terms of the density of $I_\infty(-K) = \int_0^\infty e^{-K_s} \mathrm{d}s$.*

Time to extinction

We define the first hitting time to 0 of Z as follows

$$T_0^Z = \inf\{t \geq 0, Z_t = 0\},$$

with the convention that $\inf\{\emptyset\} = \infty$. We denote by $\mathbb{P}_x$ for the law of Z starting from $x > 0$.

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$$\int_{(0,\infty)} (z \wedge z^2)\mu(\mathrm{d}z) < \infty.$$

We also assume tha that the branching mechanism ψ satisfies the so-called Grey's condition, i.e.

$$\int^{\infty} \frac{\mathrm{d}\lambda}{\psi(\lambda)} < \infty.$$

Important : Grey's condition is a necessary and sufficient condition for CB processes in random environment to be extinct with positive probability (see He et al. (2018)).

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We also introduce the CB-process in a Lévy random environment $Z^\sharp = (Z^\sharp_t, t \geq 0)$ as the unique strong solution of the following SDE

$$
\begin{aligned}
Z^\sharp_t = & Z^\sharp_0 - \psi'(0+) \int_0^t Z^\sharp_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z^\sharp_s} \mathrm{d}B_s + \int_0^t Z^\sharp_{s-} \mathrm{d}S_s \\
& + \int_0^t \int_{(0,1)} \int_0^{Z^\sharp_{s-}} z \widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z^\sharp_{s-}} z N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).
\end{aligned}
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For simplicity, we denote its law starting from $x \geq 0$ by $\mathbb{P}^{\sharp}_x$.

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Theorem

Assume that the Lévy measure associated to the branching mechanism ψ has finite first moment and g is non-decreasing. For $y \geq x \geq 0$, we have that $(Z, \mathbb{P}_x)$ is stochastically dominated by $(Z, \mathbb{P}_y)$. Moreover, the process $(Z, \mathbb{P}_x)$ is stochastically dominated by $(Z^\sharp, \mathbb{P}^\sharp_y)$.

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In particular if Grey's condition is fulfilled, then $(Z, \mathbb{P}_x)$ becomes extinct with positive probability. Furthermore if

$$\liminf_{t\to\infty} K_t = -\infty,$$

then $(Z, \mathbb{P}_x)$ becomes extinct at finite time a.s.

For our next result, we assume that following property on the competition mechanism g holds : There exists $z_0 > 0$ such that

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Theorem

Assume that the Lévy measure associated to the branching mechanism ψ has finite first moment and Grey's condition also holds together with the above assumption on g, then

$$\sup_{x \geq 0} \mathbb{E}_x \left[T_0^Z \right] < \infty.$$

If in addition $\int_{(-1,0)} \pi(\mathrm{d}z) < \infty$, the process comes down from infinity.

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Observe that under the integral condition on the competition mechanism g, the environment plays no role. In other words, it seems that the competition mechanism is too strong for the environment.

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Proposition

Let $W = (W_t, t \geq 0)$ be a standard Brownian motion independent of X and assume that g is a continuous function and non-decreasing on $[0, \infty)$ with $g(0) = 0$. For each $x > 0$, there is a unique strong solution to

$$\mathrm{d}R_t = \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\mathrm{d}X_t - \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\frac{g(R_t)}{R_t}\mathrm{d}t + \mathbf{1}_{\{R_{r-}>0:r\leq t\}}\sigma\sqrt{R_t}\mathrm{d}W_t.$$

Theorem

Let $R = (R_t, t \geq 0)$ *be as before and* $T_0^R = \sup\{s : R_s > 0\}$. *We also let* C *be the right-continuous inverse of* η, *where*

$$\eta_t = \int_0^{t \wedge T_0^R} \frac{\mathrm{d}s}{R_s}, \qquad t > 0.$$

Hence the process defined by

$$Z_t = \begin{cases} R_{C_t}, & \text{if } 0 \leq t < \eta_\infty \\ 0, & \text{if } \eta_\infty < \infty, T_0^R < \infty \text{ and } t \geq \eta_\infty, \\ \infty, & \text{if } \eta_\infty < \infty, T_0^R = \infty \text{ and } t \geq \eta_\infty \end{cases}$$

is a CSBP with competition in a Brownian random environment.

Theorem

Reciprocally, let Z CSBP with competition in a Brownian random environment with $Z_0 = x$ and let

$$C_t = \int_0^{t \wedge T_0^Z} Z_s \mathrm{d}s, \qquad t > 0.$$

If η denotes the right-continuous inverse of C, then the process defined by

$$R_t = \begin{cases} Z_{\eta_t}, & \text{if } 0 \le t < C_\infty \\ 0, & \text{if } C_\infty < \infty \text{ and } t \ge C_\infty, \end{cases}$$

satisfies the SDE from the previous definition.

Logistic case.

Bounded variation case

If X is a subordinator, then the process R satisfies

$$\mathrm{d}R_t = \mathrm{d}X_t - cR_t\mathrm{d}t + \sigma\sqrt{R_t}\mathrm{d}W_t.$$

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In other words, R is a CBI process with branching mechanism

$$\omega(\theta) = \theta\left(c + \frac{\sigma^2\theta}{2}\right) \quad \text{and} \quad \phi(\theta) = \delta\theta + \int_{(0,\infty)} (1 - e^{\theta x})\mu(\mathrm{d}x),$$

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Assume that X satisfies

$$\int_1^\infty \ln(u)\mu(\mathrm{d}u) < \infty. \tag{3.1}$$

Lemma

Let

$$m(\lambda) = -\int_0^\lambda \frac{\phi(u)}{\omega(u)}\mathrm{d}u, \qquad \textit{for} \quad \lambda \geq 0,$$

then the following identity holds

$$-m(\lambda) = \int_0^\infty \left(1 - e^{-\lambda z}\right)\frac{e^{-\frac{2c}{\sigma^2}z}}{z}\left(b + \int_0^z e^{\frac{2c}{\sigma^2}u}\overline{\mu}(u)\mathrm{d}u\right)\mathrm{d}z,$$

where $\overline{\mu}(x) = \mu(x, \infty)$*,*

Lemma

$$\delta = b - \int_{(0,1)} u\mu(\mathrm{d}u) \geq 0,$$

and the following identity

$$\int_{(0,\infty)} e^{-\lambda z}\nu(\mathrm{d}z) = e^{m(\lambda)}, \qquad \lambda \geq 0,$$

defines a unique probability measure ν *on* $(0,\infty)$ *which is infinitely divisible. In addition, it is self-decomposable if* $\overline{\mu}(0) \leq \delta$*.*

Proposition

Assume that $2\delta \geq \sigma^2 > 0$, $c > 0$ *and that* X *is a subordinator. Then the point* 0 *is polar, that is to say* $\mathbb{P}_x(T_0 < \infty) = 0$ *for all* $x > 0$. *Moreover if*

$$\int_0^1 \frac{\mathrm{d}z}{z} \exp\left\{ - \int_z^1 \int_0^\infty \frac{(1 - e^{-us})}{\omega(u)} \mu(\mathrm{d}s)\mathrm{d}u \right\} = \infty \qquad (3.2)$$

Z *is recurrent. Additionally,*

a) *If* $2\delta > \sigma^2$ *then the process* Z *is positive recurrent in* $(0, \infty)$. *Its invariant distribution* ρ *has a finite expected value if and only if the log-moment condition* (3.1) *holds. If the latter holds, then* ρ *is the size-biased distribution of* ν, *in other words*

$$\rho(\mathrm{d}z) = \left(\int_{(0,\infty)} s^{-1} \nu(\mathrm{d}s) \right)^{-1} z^{-1} \nu(\mathrm{d}z), \qquad z > 0. \quad (3.3)$$

Proposition

b) *Assume that* $2\delta = \sigma^2$ *and the log-moment condition* (3.1) *holds,*

  b.1) *if complicated integral condition is also satisfied, then* Z *is positive recurrent in* $(0, \infty)$ *and its invariant probability is defined by* (3.3),

  b.2) *or if the complicated integral condition is not satisfied, then the process* Z *is null recurrent and converges to* 0 *in probability.*

Finally, if (3.2) *is not satisfied, then* Z *is transient and for any* $x \geq a > 0$,

$$\mathbb{P}_x\left(\lim_{t\geq 0} Z_t = \infty\right) = 1 \qquad \textit{and} \qquad \mathbb{P}_x\left(\inf_{t\geq 0} Z_t < a\right) = \frac{f_0(x)}{f_0(a)}.$$

Proposition

Assume that $\sigma^2 > 2\delta$ *but* $\sigma^2 > 0$, $c > 0$ *and* X *is a subordinator, then the process converges to* 0 *with positive probability, in other words*

$$\mathbb{P}_x(\lim_{t\to 0} Z_t = 0) > 0,$$

for all $x > 0$.

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Here we want to deduce an “explicit" expression for the law of T_0^Z, the first step to reach this result is to find an explicit formulation of the function

$$G_{q,x}(\lambda) = \int_0^\infty e^{-qt}\mathbb{E}_x[e^{-\lambda Z_t}]\mathrm{d}t.$$

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$$G_{q,x}(\lambda) = \int_0^\infty e^{-qt}\mathbb{E}_x[e^{-\lambda Z_t}]\mathrm{d}t.$$

Indeed, we have that $\lim_{\lambda\to+\infty} qG_{q,x}(\lambda) = \mathbb{E}_x[e^{-qT_0}]$.

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Actually, we can do better and deduce the law of

$$T_a = \inf\{s : Z_s \leq a\},$$

for $x \geq a$.

Feller diffusion case.

In this particular case, the process satisfies

$$Z_t = Z_0 + b\int_0^t Z_s \mathrm{d}s - \int_0^t g(Z_s)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t \sigma Z_s \mathrm{d}B_s^{(e)}.$$

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Proposition

Assume that Z is the unique strong solution of the above equation, then $\mathbb{P}_x\Big(T_0^Z < \infty\Big) = 1$ accordingly as

$$\int^{\infty} \exp\left\{\int_1^{\xi} \frac{2(g(z) - bz)}{2\gamma^2 z + \sigma^2 z^2}\mathrm{d}z\right\}\mathrm{d}\xi = \infty.$$

Moreover

$$\mathbb{P}_x\left(\lim_{t\to\infty} Z_t = \infty\right) = 1 - \mathbb{P}_x\Big(T_0^Z < \infty\Big).$$

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a) if there exist $z_0 > 0$ and $w < b - \frac{\sigma^2}{2}$ such that for any $z \geq z_0$, $g(z) \leq wz$, then $\mathbb{P}_x(T_0^Z < \infty) < 1$.

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a) if there exist $z_0 > 0$ and $w < b - \frac{\sigma^2}{2}$ such that for any $z \geq z_0$, $g(z) \leq wz$, then $\mathbb{P}_x(T_0^Z < \infty) < 1$.
Example : cooperation case $g(z) \leq 0$ and $b > \frac{\sigma^2}{2}$.

b) if there exist $z_0 > 0$ and $w > b - \frac{\sigma^2}{2}$ such that for any $z \geq z_0$, $g(z) \geq wz$, then $\mathbb{P}_x(T_0^Z < \infty) = 1$.
Example : large competition $g(z) \geq bz$ for any z large enough, which is true for the logistic case $g(z) = cz^2$.

Let us define

$$T_a^Z = \inf\{t \geq 0 : Z_t \leq a\},$$

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We also introduce

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The function $S : \mathbb{R}_+ \to (0, S(+\infty))$ is continuous and bijective. In any case, we denote by $\bar{\varphi}(x)$ its inverse.

Proposition

Assume that $\gamma > 0$. *Then, for any* $x \geq a \geq 0$, *and for any* $\lambda > 0$,

$$\mathbb{E}_x\left[e^{-\lambda T_a^Z}\right] = \exp\left\{-\int_{S(a)}^{S(x)} \bar{y}_\lambda(u)\mathrm{d}u\right\}, \qquad (4.4)$$

where $\bar{y}_\lambda$ *is solution of a Ricatti equation that involves* $\bar{\varphi}$.
Note that if the integral condition of the previous proposition is satisfied, then $T_a^Z < +\infty$ *a.s.*