Regularly varying Galton–Watson processes with immigration

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Outline

Tail behaviour of

- GWI processes admitting regularly varying
  - offspring distribution
  - initial distribution
  - immigration distribution
- second order GWI processes
- stationary distribution of second order GWI processes

Galton–Watson branching process with immigration

GWI process

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i} + \varepsilon_n, \qquad n \in \mathbb{N} := \{1, 2, \ldots\},$$

where $\sum_{i=1}^{0} \xi_{n,i} := 0$,

$\{\xi_{n,i}, \varepsilon_n : n, i \in \mathbb{N}\}$ independent rv's with values in $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$,

$\{\xi_{n,i} : n, i \in \mathbb{N}\}$ identically distributed,

$\{\varepsilon_n : k \in \mathbb{N}\}$ identically distributed.

If $\varepsilon_n = 0$, $n \in \mathbb{N}$, then it is a GW process.

For notational convenience, let ξ and ε be random variables such that $\xi \stackrel{\mathcal{D}}{=} \xi_{1,1}$ and $\varepsilon \stackrel{\mathcal{D}}{=} \varepsilon_1$, and put

$$m_\xi := \mathbb{E}(\xi) \in [0, \infty], \qquad m_\varepsilon := \mathbb{E}(\varepsilon) \in [0, \infty].$$

Question

Under which conditions is the distribution of a not necessarily stationary GWI process is regularly varying at any fixed time, i.e.,

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_n > qx)}{\mathbb{P}(X_n > x)} = q^{-\alpha} \qquad \text{for all } q \in \mathbb{R}_{++} := (0, \infty)$$

for each $n \in \mathbb{N}$ with some $\alpha \in \mathbb{R}_+ := [0, \infty)$?

Regularly varying offspring distribution (BBP 2018+)

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a GWI process such that

1. ξ is regularly varying with index $\alpha \in [1, \infty)$,
2. $m_\xi \in \mathbb{R}_{++}$ in case of $\alpha = 1$,
3. there exists $r \in (\alpha, \infty)$ with $\mathbb{E}(X_0^r) < \infty$ and $\mathbb{E}(\varepsilon^r) < \infty$,
4. $\mathbb{P}(X_0 = 0) < 1$ or $\mathbb{P}(\varepsilon = 0) < 1$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim \mathbb{E}(X_0) m_\xi^{n-1} \sum_{i=0}^{n-1} m_\xi^{(\alpha-1)i} \mathbb{P}(\xi > x) + m_\varepsilon \sum_{i=1}^{n-1} m_\xi^{n-i-1} \sum_{j=0}^{n-i-1} m_\xi^{(\alpha-1)j} \mathbb{P}(\xi > x)$$

as $x \to \infty$, and hence X_n is also regularly varying with index α.

Additivity of GWI processes

If $(X_n)_{n\in\mathbb{Z}_+}$ is a GWI process, then for each $n \in \mathbb{N}$, we have

$$X_n = V^{(n)}(X_0) + \sum_{i=1}^{n} V_i^{(n-i)}(\varepsilon_i),$$

where $\{V^{(n)}(X_0), V_i^{(n-i)}(\varepsilon_i) : i \in \{1, \ldots, n\}\}$ are independent and

- $V^{(n)}(X_0)$ represents the number of individuals alive at time n, resulting from the initial individuals X_0 at time 0,
- $V_i^{(n-i)}(\varepsilon_i)$ represents the number of individuals alive at time n, resulting from the immigration ε_i at time i.

Moreover,

$$V^{(n)}(X_0) \stackrel{\mathcal{D}}{=} \sum_{\ell=1}^{X_0} \zeta_\ell^{(n)}, \qquad V_i^{(n-i)}(\varepsilon_i) \stackrel{\mathcal{D}}{=} \sum_{\ell=1}^{\varepsilon_i} \zeta_\ell^{(n-i)},$$

where $\{\zeta_\ell^{(n-i)} : i \in \{0, 1, \ldots, n\}, \ell \in \mathbb{N}\}$ are independent copies of Y_{n-i} such that $(Y_j)_{j\in\mathbb{Z}_+}$ is a GW process with initial value $Y_0 = 1$ and with the same offspring distribution as $(X_k)_{k\in\mathbb{Z}_+}$.

Moments and regular variation

1. If X is a non-negative regularly varying random variable with index $\alpha \in \mathbb{R}_{++}$, then
$$\mathbb{E}(X^\beta) \begin{cases} < \infty & \text{for all } \beta \in (-\infty, \alpha), \\ = \infty & \text{for all } \beta \in (\alpha, \infty). \end{cases}$$
2. If X and Y are non-negative random variables such that X is regularly varying with index $\alpha \in \mathbb{R}_+$ and there exists $r \in (\alpha, \infty)$ with $\mathbb{E}(Y^r) < \infty$, then
$$\mathbb{P}(Y > x) = \mathrm{o}(\mathbb{P}(X > x)) \qquad \text{as } x \to \infty.$$
3. If X_1 and X_2 are non-negative regularly varying random variables with index $\alpha_1 \in \mathbb{R}_+$ and $\alpha_2 \in \mathbb{R}_+$, respectively, such that $\alpha_1 < \alpha_2$, then
$$\mathbb{P}(X_2 > x) = \mathrm{o}(\mathbb{P}(X_1 > x)) \qquad \text{as } x \to \infty.$$

Convolution property

1. If X_1 and X_2 are non-negative random variables such that X_1 is regularly varying with index $\alpha \in \mathbb{R}_+$ and there exists $r \in (\alpha, \infty)$ with $\mathbb{E}(X_2^r) < \infty$, then
$$\mathbb{P}(X_1 + X_2 > x) \sim \mathbb{P}(X_1 > x) \qquad \text{as } x \to \infty,$$
and hence $X_1 + X_2$ is regularly varying with index α.
2. If X_1 and X_2 are independent non-negative regularly varying random variables with index $\alpha \in \mathbb{R}_+$, then
$$\mathbb{P}(X_1 + X_2 > x) \sim \mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x) \qquad \text{as } x \to \infty,$$
hence $X_1 + X_2$ is regularly varying with index α.

Regularly varying random sums (Faÿ et al. 2006, Robert and Segers 2008, Denisov, Foss and Korshunov 2010)

Let τ be a non-negative integer-valued r.v. and let $\{\zeta, \zeta_i : i \in \mathbb{N}\}$ be i.i.d. non-negative r.v., independent of τ.

1. If τ is regularly varying with index $\beta \in \mathbb{R}_+$, $\mathbb{E}(\zeta) \in \mathbb{R}_{++}$ and there exists $r \in (\beta, \infty)$ with $\mathbb{E}(\zeta^r) < \infty$, then
$$\mathbb{P}\left(\sum_{i=1}^{\tau} \zeta_i > x\right) \sim \mathbb{P}\left(\tau > \frac{x}{\mathbb{E}(\zeta)}\right) \sim (\mathbb{E}(\zeta))^\beta \, \mathbb{P}(\tau > x) \qquad \text{as } x \to \infty.$$
2. If ζ is regularly varying with index $\alpha \in [1, \infty)$, $\mathbb{E}(\zeta) \in \mathbb{R}_{++}$, $\mathbb{P}(\tau = 0) < 1$ and there exists $r \in (\alpha, \infty)$ with $\mathbb{E}(\tau^r) < \infty$, then
$$\mathbb{P}\left(\sum_{i=1}^{\tau} \zeta_i > x\right) \sim \mathbb{E}(\tau)\, \mathbb{P}(\zeta > x) \qquad \text{as } x \to \infty.$$
3. If τ and ζ are regularly varying with index $\beta \in [1, \infty)$, $\mathbb{P}(\zeta > x) = \mathrm{O}(\mathbb{P}(\tau > x))$ as $x \to \infty$ and $\mathbb{E}(\tau), \mathbb{E}(\zeta) \in \mathbb{R}_{++}$, then
$$\mathbb{P}\left(\sum_{i=1}^{\tau} \zeta_i > x\right) \sim \mathbb{E}(\tau)\, \mathbb{P}(\zeta > x) + (\mathbb{E}(\zeta))^\beta \, \mathbb{P}(\tau > x) \qquad \text{as } x \to \infty.$$

Sketch of proof

- By the additivity:

$$X_n \stackrel{\mathcal{D}}{=} V^{(n)}(X_0) + \sum_{i=1}^{n} V_i^{(n-i)}(\varepsilon_i)$$

with

$$V^{(n)}(X_0) \stackrel{\mathcal{D}}{=} \sum_{\ell=1}^{X_0} \zeta_\ell^{(n)}, \qquad V_i^{(n-i)}(\varepsilon_i) \stackrel{\mathcal{D}}{=} \sum_{\ell=1}^{\varepsilon_i} \zeta_\ell^{(n-i)},$$

where $\{\zeta_\ell^{(n-i)} : i \in \{0, 1, \ldots, n\}, \ell \in \mathbb{N}\}$ are independent copies of Y_{n-i} such that $(Y_j)_{j\in\mathbb{Z}_+}$ is a GW process with initial value $Y_0 = 1$ and with the same offspring distribution as $(X_k)_{k\in\mathbb{Z}_+}$.

- We prove $\mathbb{P}(Y_j > x) \sim m_\xi^{j-1} \sum_{i=0}^{j-1} m_\xi^{(\alpha-1)i} \, \mathbb{P}(\xi > x)$ by induction:
  - For $j = 1$ obvious, since $Y_1 = \xi_{1,1}$.
  - Induction hypothesis: the statement holds for $\{1, \ldots, j-1\}$.
  - By the Markov property, we have $Y_j \stackrel{\mathcal{D}}{=} V^{(j-1)}(\xi_{1,1})$, where $(V^{(k)}(\xi_{1,1}))_{k\in\mathbb{Z}_+}$ is GW process with initial value $V^{(0)}(\xi_{1,1}) = \xi_{1,1}$.
  - By the additivity: $V^{(j-1)}(\xi_{1,1}) \stackrel{\mathcal{D}}{=} \sum_{i=1}^{\xi_{1,1}} \zeta_i^{(j-1)}$, where $\{\zeta_i^{(j-1)} : i \in \mathbb{N}\}$ are independent copies of Y_{j-1} such that $\{\xi_{1,1}, \zeta_i^{(j-1)} : i \in \mathbb{N}\}$ are independent.
  - By the induction hypothesis: $\mathbb{P}(\zeta_i^{(j-1)} > x) = \mathbb{P}(Y_{j-1} > x) = \mathrm{O}(\mathbb{P}(\xi > x))$ as $x \to \infty$, thus by the regularly varying random sums and the induction hypothesis,

$$\mathbb{P}(Y_j > x) = \mathbb{P}\left(\sum_{i=1}^{\xi_{1,1}} \zeta_i^{(j-1)} > x\right) \sim \mathbb{E}(\xi_{1,1}) \, \mathbb{P}(\zeta_1^{(j-1)} > x) + (\mathbb{E}(\zeta_1^{(j-1)}))^\alpha \, \mathbb{P}(\xi_{1,1} > x)$$
$$\sim m_\xi \, \mathbb{P}(Y_{j-1} > x) + m_\xi^{(j-1)\alpha} \, \mathbb{P}(\xi > x) \sim m_\xi^{j-1} \sum_{i=0}^{j-1} m_\xi^{(\alpha-1)i} \, \mathbb{P}(\xi > x).$$

- By the regularly varying random sums,

$$\mathbb{P}(V_i^{(n-i)}(\varepsilon_i) > x) \sim \mathbb{E}(\xi_{1,1}) \, \mathbb{P}(\zeta_1^{(n-i)} > x) \sim m_\varepsilon \, \mathbb{P}(Y_{n-i} > x).$$

Regularly varying initial distribution (BBP 2018+)

Let $(X_n)_{n \in \mathbb{Z}_+}$ be a GWI process such that

1. X_0 is regularly varying with index $\beta \in \mathbb{R}_+$,
2. $\mathbb{P}(\xi = 0) < 1$,
3. there exists $r \in (1 \vee \beta, \infty)$ with $\mathbb{E}(\xi^r) < \infty$ and $\mathbb{E}(\varepsilon^r) < \infty$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim m_\xi^{n\beta} \, \mathbb{P}(X_0 > x) \qquad \text{as } x \to \infty,$$

and hence X_n is also regularly varying with index β.

Regularly varying immigration distribution (BBP 2018+)

Let $(X_n)_{n \in \mathbb{Z}_+}$ be a GWI process such that

1. ε is regularly varying with index $\gamma \in \mathbb{R}_+$,
2. $\mathbb{P}(\xi = 0) < 1$,
3. there exists $r \in (1 \vee \gamma, \infty)$ with $\mathbb{E}(\xi^r) < \infty$ and $\mathbb{E}(X_0^r) < \infty$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim \sum_{i=1}^{n} m_\xi^{(n-i)\gamma} \, \mathbb{P}(\varepsilon > x) \qquad \text{as } x \to \infty,$$

and hence X_n is also regularly varying with index γ.

Regularly varying offspring and initial distributions (BBP 2018+)

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a GWI process such that

1. X_0 and ξ are regularly varying with index $\alpha \in [1, \infty)$,
2. $\mathbb{E}(X_0) \in \mathbb{R}_{++}$ and $m_\xi \in \mathbb{R}_{++}$ in case of $\alpha = 1$,
3. $\mathbb{P}(\xi > x) = \mathrm{O}(\mathbb{P}(X_0 > x))$ as $x \to \infty$,
4. there exists $r \in (\alpha, \infty)$ such that $\mathbb{E}(\varepsilon^r) < \infty$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim \mathbb{E}(X_0) m_\xi^{n-1} \sum_{i=0}^{n-1} m_\xi^{(\alpha-1)i} \, \mathbb{P}(\xi > x) + m_\xi^{n\alpha} \, \mathbb{P}(X_0 > x)$$
$$+ m_\varepsilon \sum_{i=1}^{n-1} m_\xi^{n-i-1} \sum_{j=0}^{n-i-1} m_\xi^{(\alpha-1)j} \, \mathbb{P}(\xi > x)$$

as $x \to \infty$, and hence X_n is also regularly varying with index α.

Regularly varying offspring and immigration (BBP 2018+)

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a GWI process such that

1. ξ and ε are regularly varying with index $\alpha \in [1, \infty)$,
2. $m_\xi \in \mathbb{R}_{++}$ and $m_\varepsilon \in \mathbb{R}_{++}$ in case of $\alpha = 1$,
3. $\mathbb{P}(\xi > x) = \mathrm{O}(\mathbb{P}(\varepsilon > x))$ as $x \to \infty$,
4. there exists $r \in (\gamma, \infty)$ with $\mathbb{E}(X_0^r) < \infty$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim \mathbb{E}(X_0) m_\xi^{n-1} \sum_{i=0}^{n-1} m_\xi^{(\alpha-1)i}\, \mathbb{P}(\xi > x) + m_\varepsilon \sum_{j=1}^{n-1} m_\xi^{n-j-1} \sum_{i=0}^{n-j-1} m_\xi^{(\alpha-1)i}\, \mathbb{P}(\xi > x) + \sum_{j=1}^{n} m_\xi^{(n-j)\alpha}\, \mathbb{P}(\varepsilon > x)$$

as $x \to \infty$, and hence X_n is also regularly varying with index α.

Regularly varying initial value and immigration (BBP 2018+)

Let $(X_n)_{n \in \mathbb{Z}_+}$ be a GWI process such that

1. X_0 and ε are regularly varying with index $\beta \in \mathbb{R}_+$,
2. $\mathbb{P}(\xi = 0) < 1$,
3. there exists $r \in (1 \vee \beta, \infty)$ with $\mathbb{E}(\xi^r) < \infty$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim m_\xi^{n\beta} \, \mathbb{P}(X_0 > x) + \sum_{i=1}^{n} m_\xi^{(n-i)\beta} \, \mathbb{P}(\varepsilon > x) \qquad \text{as } x \to \infty,$$

and hence X_n is also regularly varying with index β.

Regularly varying initial, offspring and immigration distributions (BBP 2018+)

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a GWI process such that

1. X_0, ξ and ε are regularly varying with index $\alpha \in [1, \infty)$,
2. $\mathbb{E}(X_0) \in \mathbb{R}_{++}$, $m_\xi \in \mathbb{R}_{++}$ and $m_\varepsilon \in \mathbb{R}_+$ in case of $\alpha = 1$,
3. $\mathbb{P}(\xi > x) = \mathrm{O}(\mathbb{P}(X_0 > x))$ as $x \to \infty$ and
   $\mathbb{P}(\xi > x) = \mathrm{O}(\mathbb{P}(\varepsilon > x))$ as $x \to \infty$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim \mathbb{E}(X_0) m_\xi^{n-1} \sum_{i=0}^{n-1} m_\xi^{(\alpha-1)i} \, \mathbb{P}(\xi > x) + m_\xi^{n\alpha} \, \mathbb{P}(X_0 > x)$$

$$+ m_\varepsilon \sum_{j=1}^{n-1} m_\xi^{n-j-1} \sum_{i=0}^{n-j-1} m_\xi^{(\alpha-1)i} \, \mathbb{P}(\xi > x) + \sum_{j=1}^{n} m_\xi^{(n-j)\alpha} \, \mathbb{P}(\varepsilon > x)$$

as $x \to \infty$, and hence X_n is also regularly varying with index α.

Second order GWI process

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i} + \sum_{i=1}^{X_{n-2}} \eta_{n,i} + \varepsilon_n, \qquad n \in \mathbb{N},$$

where $\{\xi_{n,i}, \eta_{n,i}, \varepsilon_n : n, i \in \mathbb{N}\}$ independent rv's with values in $\mathbb{Z}_+$,

$\{\xi_{n,i} : n, i \in \mathbb{N}\}$ identically distributed,

$\{\eta_{n,i} : n, i \in \mathbb{N}\}$ identically distributed,

$\{\varepsilon_n : k \in \mathbb{N}\}$ identically distributed.

$(X_n)_{n \geqslant -1}$ is a second order Markov chain

2-type representation

Put

$$\boldsymbol{Z}_n := \begin{bmatrix} Z_{n,1} \\ Z_{n,2} \end{bmatrix} := \begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix}, \qquad n \in \mathbb{Z}_+.$$

This yields

$$\boldsymbol{Z}_n = \sum_{i=1}^{Z_{n-1,1}} \begin{bmatrix} \xi_{n,i} \\ 1 \end{bmatrix} + \sum_{i=1}^{Z_{n-1,2}} \begin{bmatrix} \eta_{n,i} \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_n \\ 0 \end{bmatrix}, \qquad n \in \mathbb{N},$$

hence $(\boldsymbol{Z}_n)_{n\in\mathbb{Z}_+}$ is a 2-type GWI process with initial vector $\boldsymbol{Z}_0 = \begin{bmatrix} X_0 \\ X_{-1} \end{bmatrix}$, which is a Markov chain.

Offspring mean matrix:

$$\boldsymbol{M}_{\xi,\eta} := \begin{bmatrix} m_\xi & m_\eta \\ 1 & 0 \end{bmatrix}$$

with $m_\xi := \mathbb{E}(\xi_{1,1}) \in [0,\infty]$ and $m_\eta := \mathbb{E}(\eta_{1,1}) \in [0,\infty]$.

Expectation of a second order GW process

If $(Y_n)_{n\geqslant -1}$ is a second-order GW process with $m_\xi, m_\eta \in \mathbb{R}_+$ satisfying $m_\xi + m_\eta > 0$ and with initial values $Y_0 = 1$, $Y_{-1} = 0$, then

$$m_n := \mathbb{E}(Y_n) = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-}, \qquad n \in \mathbb{N},$$

where

$$\lambda_+ := \frac{m_\xi + \sqrt{m_\xi^2 + 4m_\eta}}{2} \in \mathbb{R}_{++}, \quad \lambda_- := \frac{m_\xi - \sqrt{m_\xi^2 + 4m_\eta}}{2} \in (-\lambda_+, 0]$$

are the eigenvalues of $\boldsymbol{M}_{\xi,\eta}$.

Indeed, $\mathbb{E}(Y_n) = m_\xi \, \mathbb{E}(Y_{n-1}) + m_\eta \, \mathbb{E}(Y_{n-2})$, which can be written in the matrix form

$$\begin{bmatrix} \mathbb{E}(Y_n) \\ \mathbb{E}(Y_{n-1}) \end{bmatrix} = \boldsymbol{M}_{\xi,\eta} \begin{bmatrix} \mathbb{E}(Y_{n-1}) \\ \mathbb{E}(Y_{n-2}) \end{bmatrix} \quad \text{implying} \quad \begin{bmatrix} \mathbb{E}(Y_n) \\ \mathbb{E}(Y_{n-1}) \end{bmatrix} = \boldsymbol{M}_{\xi,\eta}^n \begin{bmatrix} \mathbb{E}(Y_0) \\ \mathbb{E}(Y_{-1}) \end{bmatrix}.$$

Regularly varying initial, offspring and immigration distributions (Bősze and P 2018+)

Let $(X_n)_{n \geqslant -1}$ be a second order GWI process such that

1. X_0, X_{-1}, ξ, η and ε are regularly varying with index $\alpha \in [1, \infty)$,
2. $\mathbb{E}(X_0), \mathbb{E}(X_{-1}), m_\xi, m_\varepsilon \in \mathbb{R}_{++}$ and $m_\eta \in \mathbb{R}_+$,
3. $\mathbb{P}(\xi > x) = \mathrm{O}(\mathbb{P}(X_0 > x))$, $\mathbb{P}(\xi > x) = \mathrm{O}(\mathbb{P}(X_{-1} > x))$, $\mathbb{P}(\eta > x) = \mathrm{O}(\mathbb{P}(X_0 > x))$, $\mathbb{P}(\eta > x) = \mathrm{O}(\mathbb{P}(X_{-1} > x))$, $\mathbb{P}(\xi > x) = \mathrm{O}(\mathbb{P}(\varepsilon > x))$ and $\mathbb{P}(\eta > x) = \mathrm{O}(\mathbb{P}(\varepsilon > x))$ as $x \to \infty$.

Then for each $n \in \mathbb{N}$, we have

$$\mathbb{P}(X_n > x) \sim \begin{bmatrix} \mathbb{E}(X_0) \\ \mathbb{E}(X_{-1}) \end{bmatrix}^\top \sum_{k=0}^{n-1} m_k^\alpha \left(\boldsymbol{M}_{\xi,\eta}^{n-k-1}\right)^\top \begin{bmatrix} \mathbb{P}(\xi > x) \\ \mathbb{P}(\eta > x) \end{bmatrix}$$
$$+ \begin{bmatrix} m_\varepsilon \\ 0 \end{bmatrix}^\top \sum_{i=1}^{n-1} \sum_{j=0}^{n-i-1} m_j^\alpha \left(\boldsymbol{M}_{\xi,\eta}^{n-j-1}\right)^\top \begin{bmatrix} \mathbb{P}(\xi > x) \\ \mathbb{P}(\eta > x) \end{bmatrix}$$
$$+ m_n^\beta \, \mathbb{P}(X_0 > x) + m_{n-1}^\alpha m_\eta^\alpha \, \mathbb{P}(X_{-1} > x) + \sum_{i=1}^{n} m_{n-i}^\alpha \, \mathbb{P}(\varepsilon > x).$$

Tail behavior of the stationary distribution of a GWI (Basrak, Kulik and Palmowski 2013)

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a GWI process such that

1. $m_\xi \in (0, 1)$,
2. ε is regularly varying with index $\alpha \in (0, 2)$,
3. $\mathbb{E}(\xi^2) < \infty$ in case of $\alpha \in [1, 2)$.

Then the tail of the unique stationary distribution μ of $(X_n)_{n\in\mathbb{Z}_+}$ satisfies

$$\mu((x, \infty)) \sim \sum_{i=0}^{\infty} m_\xi^{i\alpha}\, \mathbb{P}(\varepsilon > x) = \frac{\mathbb{P}(\varepsilon > x)}{1 - m_\xi^{\alpha}} \qquad \text{as } x \to \infty,$$

and hence μ is also regularly varying with index α.

- The above result is valid also for $\alpha \in [2, 3)$ under the additional assumption $\mathbb{E}(\xi^3) < \infty$.
- The same seems to apply also for $\alpha \in [3, \infty)$ possibly under the additional assumption $\mathbb{E}(\xi^{\lfloor\alpha\rfloor+1}) < \infty$.

Stationary distribution of the 2-type representation of a second order GWI process

Let $(X_n)_{n\geqslant -1}$ be a second order GWI process such that

1. $m_\xi \in \mathbb{R}_{++}$, $m_\eta \in \mathbb{R}_{++}$, $m_\xi + m_\eta < 1$,
2. $\mathbb{P}(\varepsilon = 0) < 1$ and $\mathbb{E}(\mathbb{1}_{\{\varepsilon \neq 0\}} \log(\varepsilon)) < \infty$.

Then there exists a unique stationary distribution π for the Markov chain

$$\left(\begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} \right)_{n \in \mathbb{Z}_+}.$$

Moreover, the marginals of π are the same distributions μ, admitting the representation

$$\mu \stackrel{\mathcal{D}}{=} \sum_{i=0}^{\infty} V_i^{(i)}(\varepsilon_i),$$

where $(V_k^{(i)}(\varepsilon_i))_{k\geqslant -1}$, $i \in \mathbb{Z}_+$, are independent copies of $(Y_k(\varepsilon))_{k\geqslant -1}$, which is a second-order GW process with initial values $Y_0(\varepsilon) = \varepsilon$ and $Y_{-1}(\varepsilon) = 0$, and with the same offspring distributions as $(X_k)_{k\geqslant -1}$.

Tail behavior of the stationary distribution of the 2-type representations of second order GWI processes (BBP 2018+)

Let $(X_n)_{n\in\mathbb{Z}_+}$ be a second order GWI process such that

1. $m_\xi \in \mathbb{R}_{++}$, $m_\eta \in \mathbb{R}_{++}$, $m_\xi + m_\eta < 1$,
2. ε is regularly varying with index $\alpha \in (0, 2)$,
3. $\mathbb{E}(\xi^2) < \infty$ and $\mathbb{E}(\eta^2) < \infty$ in case of $\alpha \in [1, 2)$.

Then the tail of the marginals μ of the unique stationary distribution π of the Markov chain

$$\left(\begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix}\right)_{n\in\mathbb{Z}_+}$$

satisfies

$$\mu((x,\infty)) \sim \sum_{i=0}^{\infty} m_i^\alpha \, \mathbb{P}(\varepsilon > x) \qquad \text{as } x \to \infty,$$

and hence μ is also regularly varying with index α.

Further remarks and open questions

- The tail behavior of the stationary distribution of the 2-type representation of a second order GWI process is the limit as $n \to \infty$ of the corresponding tail behavior of non-stationary processes.
- In all results for second order processes, if we assume that $\eta \equiv 0$, we get back formally the results for first order processes.
- The same techniques can be used for higher order GWI processes.
- Joint regular variation for (general) 2-type GWI processes?

References

BARCZY, M., BŐSZE, ZS. and PAP, G. (2018+).
Regularly varying non-stationary Galton–Watson processes with immigration. To appear in: *Statistics & Probability Letters*
http://arxiv.org/abs/1801.04002

BARCZY, M., BŐSZE, ZS. and PAP, G. (2018+).
On tail behavior of stationary second-order Galton–Watson processes with immigration.
http://arxiv.org/abs/1801.07931

BASRAK, B., KULIK, R. and PALMOWSKI, Z. (2013).
Heavy-tailed branching process with immigration. *Stochastic Models* **29(4)** 413–434.

WACHTEL, V. I., DENISOV, D. and KORSHUNOV, D. (2013).
Tail asymptotics for the supercritical Galton–Watson process in the heavy-tailed case. *Proceedings of the Steklov Institute of Mathematics* **282** 273–297.

References

DENISOV, D., FOSS, S. and KORSHUNOV, D. (2010).
Asymptotics of randomly stopped sums in the presence of heavy tails. *Bernoulli* **16(4)** 971–994.

FAŸ, G., GONZÁLEZ-ARÉVALO, B., MIKOSCH, T. and SAMORODNITSKY, G. (2006).
Modeling teletraffic arrivals by a Poisson cluster process. *Queueing Systems* **54** 121–140.

ROBERT, C. Y. and SEGERS, J. (2008)
Tails of random sums of a heavy-tailed number of light-tailed terms. *Insurance: Mathematics and Economics* **43** 85–92.

EMBRECHTS, P., KLÜPPELBERG, C. and MIKOSCH, T. (1997).
Modelling Extremal Events for Insurance and Finance. Springer, Berlin.

Thank you for your attention!