Financial market model based on randomly indexed branching processes and Hawkes processes

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The purpose of this talk is to present an alternative model for the price of risky financial assets and some considerations about of pricing of simple financial derivatives associated, based on Epps T.W. (1996) and Mitov G. and Mitov K. (2007).

Standard Financial Market Model:

Given $(\Omega, \mathcal{F}, P)$, consider a financial market consisting of two assets: a risk free asset with price process $B(t)$, and a stock with price process $S(t)$.

- The price process $B(t)$ has the dynamics:

$$dB(t) = r(t)B(t)dt$$

where $r(t)$ is any adapted process. A natural interpretation of a risk free asset is that it corresponds to a bank with the (possibly stochastic) short rate of interest $r(t)$.

- The stock price $S(t)$ is given by:

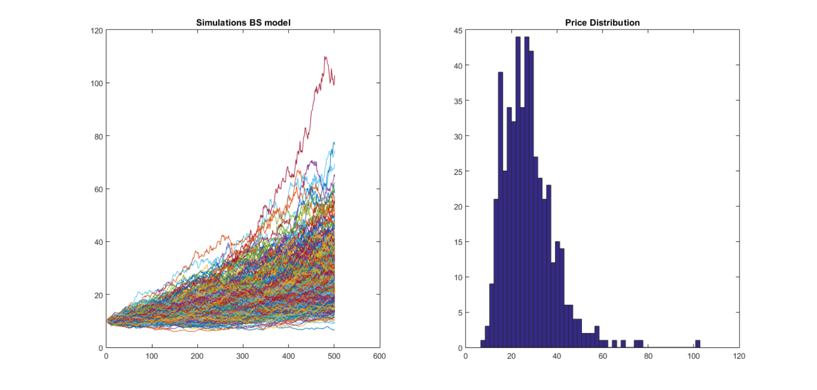
$$dS(t) = S(t)\mu(t, S(t))dt + S(t)\sigma(t, S(t))dW(t)$$

where $W(t)$ is a Wiener process, μ is the local mean rate of return and σ as the volatility of $S(t)$.

The most important special case of this model occurs when r, μ and σ are deterministic constants. This is the famous **Black-Scholes model**.

$$dB(t) = rB(t)dt \quad \Rightarrow \quad B(t) = B(0)e^{rt}$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad \Rightarrow \quad S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$


Simulations BS model
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A financial derivative with date of maturity T, is other financial asset of the form $V(t) = \Phi(t, S(t))$.(Ex: Call and Put options, Futures, Forwards, Swaps,...)

$$¿V(t) = \Phi(t, S(t))?$$

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$$¿V(t) = \Phi(t, S(t))?$$

The main assumption we will make is that the market is **efficient** in the sense that it is **free of arbitrage possibilities**. An arbitrage possibility is a self-financed portfolio $h(t)$ such that:

$$h(0) = 0 \quad ; \quad P[h(t) \geqslant 0] = 1 \quad ; \quad P[h(t) > 0] > 0$$

(The First Fundamental Theorem) The model is arbitrage free essentially if and only if there exists a (local) martingale measure $\mathbb{Q}$.

A probability measure $\mathbb{Q}$ is called an equivalent martingale measure for the market model on the time interval $[0, T]$, if it has the following properties:

- $\mathbb{Q}$ is equivalent to $\mathbb{P}$.
- All price processes are martingales under $\mathbb{Q}$ on the time interval $[0, T]$.

For determining a reasonable price process $V(t)$ are two main approaches:

- The derivative should be priced in a way that is consistent with the price of the underlying assets. More precisely we should demand that the extended market $\{V(t), B(t), S(t)\}$ is free of arbitrage possibilities.
- If the derivative is attainable, with hedging portfolio Π, then the only reasonable price is given by $\Pi(t) = V(t)$.

(General Pricing Formula) The arbitrage free price process for the derivative $V(t)$ is given by:

$$V(t) = B(t)E^{Q}\left[\frac{V(T)}{B(T)}\middle|\mathcal{F}_t\right]$$

where Q is the (not necessarily unique) martingale measure.

Assuming the existence of a short rate, the pricing formula takes the form:

$$V(t) = E^{Q}\left[e^{-\int_t^T r(s)ds}V(T)\middle|\mathcal{F}_t\right]$$

In the case of the Black-Scholes model:

$$dB(t) = rB(t)dt$$
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

and the Q-dynamics of S are given by:

$$dS(t) = rS(t)dt + \sigma S(t)d\hat{W}(t)$$

We can write $S(T)$ explicitly as:

$$S(T) = S(t)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(\hat{W}(T) - \hat{W}(t))\right\}$$

Thus we have the pricing formula:

$$V(t) = E^{Q}\left[e^{-\int_t^T r(s)ds}V(T)\middle|\mathcal{F}_t\right] = e^{-r(T-t)}\int_{-\infty}^{\infty}\Phi(S(t)e^{X})f(x)dx$$

where f is the density of a random variable X with the distribution

$$N\left[\left(r-\frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right]$$

Some drawbacks in BS

- Extensive empirical evidence, documented that the logarithm of stock return tend to be leptokurtic; that is, their distributions have thicker tails than the normal distribution derived from the geometric Brownian motion law.
- The stock prices sometimes exhibit large jumps when some important news is disclosed.
- **The stock prices were quoted in units of minimum tick size in many markets. This discreteness of the stock price contradicts the continuous distribution assumption in the Black-Scholes**.

Randomly indexed branching process as a price process

At 1996 T.W. Epps introduced a randomly indexed branching process for modeling the stock price. The model is constructed by Bienayme-Galton-Watson branching process subordinated with a Poisson process. has the following features:

- The extra randomness introduced by the subordination produces in the increments and in the returns the same high proportion of outliers observed in high-frequency stock data.
- The model predicts an inverse relation between variance of returns and the initial price which is well documented empirically.
- The possibility of extinction of the stock price process have natural interpretation as bankruptcy.

Taking $S_0 > 0$,

- For each n, let $K_{0n} = 0$ and $\{K_{jn}\}_{j \in \mathbb{N}}$ to be i.i.d., nonnegative, integer-valued random variables.
- Define $\{S_n\}_{n=0}^{\infty}$ as:

$$S_n = \sum_{j=0}^{S_{n-1}} K_{jn}$$

- Set $N_0 = 0$ and introduce a nondecreasing, integer-valued process $\{N_t\}_{t \geqslant 0}$ independent of the $\{K_{jn}\}$, and having stationary, independent increments.

Taking $S(0) \equiv S_0$ and

$$\{S(t)\}_{t \geqslant 0} := \{S_{N_t}\}_{t \geqslant 0}$$

delivers an integer-valued process that evolves in continuous time.

Here $S(t)$ represents the price of one share of stock at time t measured in units of minimum price movements (for example \$0.01). Equity prices are then viewed as consisting of an integer number of *price particles*. In each period, each *price particle* of equity price produces a random number of offspring *price particles*, the aggregate number of which comprises the equity price in the next period.

Remarks:

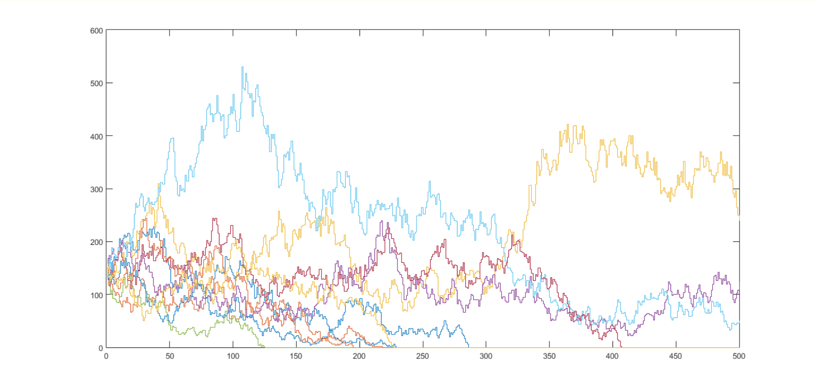
- In this application $\{N_t\}_{t \geqslant 0}$ can be thought of as counting the number of information events up to t.
- The price process is pure jump, since $S(t)$ does not change between information shocks.
- $S(t) - S(t-)$ is always an integral multiple of the minimum tick size whenever branching does occur.
- The model also implies a positive probability of extinction ($S(t) = 0$).

- Letting $p_k = \mathbb{P}[K = k]$ for $k = 0, 1, 2, \ldots$ represent the probability mass function of K and $f_K(t) = \sum_{k=0}^{\infty} t^k p_k \; S(t)$ the p.g.f., we have for the generating function of S_n (the price after n shocks) conditional on S_{n-1}:

$$f_{S_n}(t|S_{n-1}) = E[t^{S_n}|S_{n-1}] = E[t^{\sum_{j=0}^{S_{n-1}} K_{jn}}|S_{n-1}] = (f_K(t))^{S_{n-1}}$$

- The generating function of $S(t) \equiv S_{N_t}$ is then

$$f_{S(t)}(t) = \sum_{n=0}^{\infty} \left(f_K^{[n]}(t) \right)^{S_0} \mathbb{P}[N_t = n]$$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- Let us consider a Bienayme-Galton-Watson branching process, $\{S_n\}_{n=0}^{\infty}$ with a non-random number of ancestors $S_0 > 0$ and the offspring probability distribution is :

$$\mathbb{P}[S_{n+1} = 0|S_n = 1] = 1 - a$$
$$\mathbb{P}[S_{n+1} = k|S_n = 1] = ap(1-p)^{k-1} \ , \ k = 1, 2, \dots$$

where $0 < a < 1$ y $0 < p < 1$.(The two-parameter geometric distribution).

- The probability generating function :

$$f_K(t) = 1 - \frac{m(1-t)}{1 + \frac{b}{2m}(1-t)}, \quad t \in [0,1]$$

where $m = a/p$. Then,

$$f_K^{[n]}(t) = 1 - \frac{m^n(1-t)}{1 + \frac{b}{2m}\frac{1-m^n}{1-m}(1-t)}$$

- The mean and variance are:

$$\mu = 2\frac{1-p}{p}m \quad ; \quad \sigma^2 = \frac{a}{p}\left(\frac{(1-p)+(1-a)}{p}\right)$$

- Differentiating:

$$\frac{d^{(p)}(f_K^{[n]}(t))}{dt^p} = \frac{p!m^n \left[\frac{b(1-m)^n}{2m(1-m)}\right]^{p-1}}{\left[1+\frac{b(1-m^n)}{2m(1-m)}(1-t)\right]^{p+1}}$$

and follows that,

$$\mathbb{P}[S_n = k|S_0 = 1] = \frac{m^n \left[\frac{b(1-m)^n}{2m(1-m)}\right]^{p-1}}{\left[1+\frac{b(1-m^n)}{2m(1-m)}\right]^{p+1}}$$

- Consider an independent of $\{S_n\}$, Poisson process $\{N_t\}_{t\geqslant 0}$ with constant intensity $\lambda > 0$. Define the randomly indexed branching process $\{S(t)\}_{t\geqslant 0} = \{S_{N_t}\}$.
- Starting with $S(0) \geqslant 1$ ancestors, the p.g.f. of the process $S(t)$ is:

$$
\begin{aligned}
\Phi(u,t) &= \sum_{n=0}^{\infty} \frac{(\lambda u)^n}{n!} e^{-\lambda u} (f_K^{[n]}(t))^{S(0)} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda u)^n}{n!} e^{-\lambda u} \left(1 - \frac{m^n(1-t)}{1 + \frac{b}{2m}\frac{1-m^n}{1-m}(1-t)}\right)^{S(0)}
\end{aligned}
$$

- The formulas for the mean and the variance of the process $S(t)$ are:

$$M(t) = E[S(t)|S(0)] = \begin{cases} S(0)e^{\lambda t(m-1)} & \text{si } m \neq 1 \\ S(0) & \text{si } m = 1 \end{cases}$$

$$\sigma^2(t) = Var[S(t)|S(0)] = \begin{cases} S(0)^2[e^{\lambda t(m^2-1)} - e^{2\lambda t(m-1)}] + \\ \quad \frac{S(0)\sigma^2[e^{\lambda t(m^2-1)} - e^{\lambda t(m-1)}]}{m(m-1)} & \text{si } m \neq 1 \\ S(0)\sigma^2\lambda t & \text{si } m = 1 \end{cases}$$

Under the conditions assumed above:

1. The process $S(t)m^{-N_t}; t \geqslant 0$ is a nonnegative martingale.
2. The process $S(t)e^{-\lambda t(m-1)}; t \geqslant 0$ is a nonnegative martingale.
3. It follows that the discounted stock price process $S(t)e^{-rt}$, has mean

$$E[S(t)e^{-rt}|S(0)] = e^{[-\lambda(m-1)-r]t}S(0)$$

We can state that discounted stock price process $S(t)e^{-rt}$ will be a martingale if the parameters of the distribution of $S(t)$ are such that:

$$\lambda(m-1) = r \Leftrightarrow \lambda\frac{a-p}{p} = r \Leftrightarrow a = p(1+r/\lambda)$$

Utilizing this relation we define EMM $\mathbb{Q}$ as follows:

- $\mathbb{Q}$ to be equal to the real measure P on the elementary sets of the Poisson process N_t.
- We define $\mathbb{Q}$ on the elementary sets of the branching process by:

$$\mathbb{Q}[S_{n+1}=0|S_n=1] = 1-\hat{a}$$
$$\mathbb{Q}[S_{n+1}=k|S_n=1] = \hat{a}p(1-p)^{k-1}\ ,\ k=1,2,\ldots$$

where $\hat{a}$ satisfies the initial condition.

Call options pricing

$$
\begin{aligned}
C(T;0) &= e^{-rT}E^{\mathbb{Q}}[(S(T)-K)^{+}|S(0)] \\
&= e^{-r(T-t)}\left[\sum_{k=K+1}^{\infty} k\mathbb{Q}[S(T)=k|S(0)] - K\mathbb{Q}[S(T)>K|S(0)]\right] \\
&= e^{-r(T-t)}\left[E^{\mathbb{Q}}[S(T)|S(0)] - K\right] \\
&+ e^{-r(T-t)}\left[K\mathbb{Q}[S(T)\leqslant K|S(0)] - \sum_{k=1}^{K} k\mathbb{Q}[S(T)=k|S(0)]\right]
\end{aligned}
$$

Using the relation:

$$\mathbb{Q}[S(T) = k|S(0)] = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} \mathbb{Q}[S_n = k|S_0]$$

and the fact that $S(t)e^{-\lambda t(m-1)}$ is a nonnegative martingale

$$C(T;0) = S(0) - e^{-rT}K + e^{-(r+\lambda)T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \sum_{k=0}^{K} (K-k)\mathbb{Q}(S_n = k|S_0)$$

Hawkes process

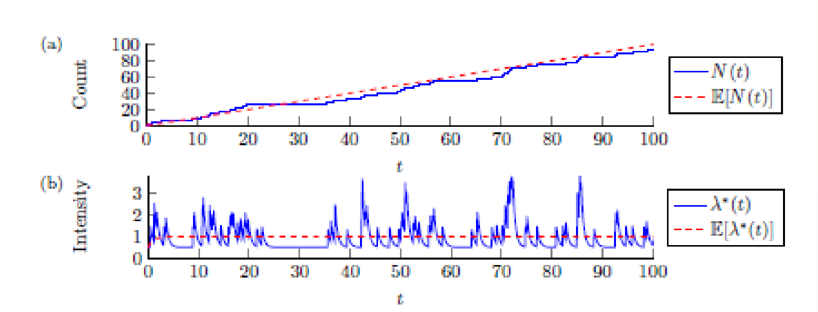
The research proposal consider that the process N_t in the randomly indexed branching processes model for price, is defined considering that the processes $\{N_t\}_{t\geqslant 0}$ that satisfies

$$P[N(t+h)-N(t)=m|\mathcal{F}_t]=\begin{cases}\lambda(t)h+o(h) & m=1\\ o(h) & m>1\\ 1-\lambda(t)h+o(h) & m=0\end{cases}$$

with conditinal intensity function

$$\lambda(t)=\alpha+\int_0^t \mu(t-u)dN(u)$$

for some $\alpha>0$ (background intensity) and $\mu:(0,\infty)\to[0,\infty)$ (excitation function).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