A note on branching processes in varying environment with generation-dependent immigration

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Joint work with Miguel González, Götz Kersting and Inés del Puerto

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Department of Mathematics
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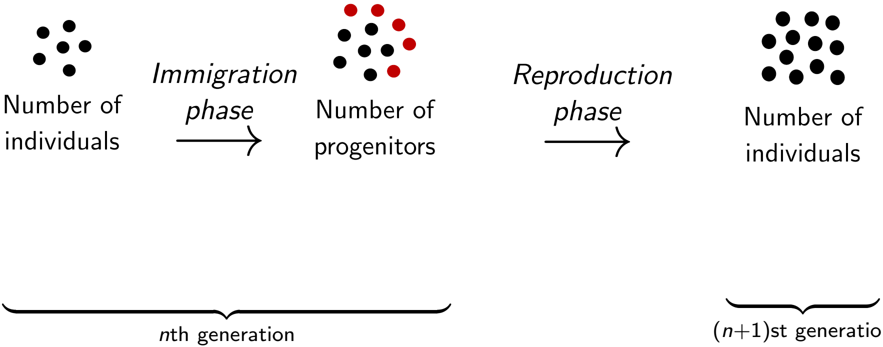
University of Extremadura
Spain

Contents

González, G., Kersting, G., Minuesa, C., del Puerto, I. (2018)
Branching processes in varying environment with generation-dependent immigration. *Work in progress*.

- Discrete-time stochastic model (non-overlapping generations).
- There is an immigration process in each generation.
- 2 phases: { Immigration phase. / Reproduction phase.
- The distributions that govern both phases depend on the generation.



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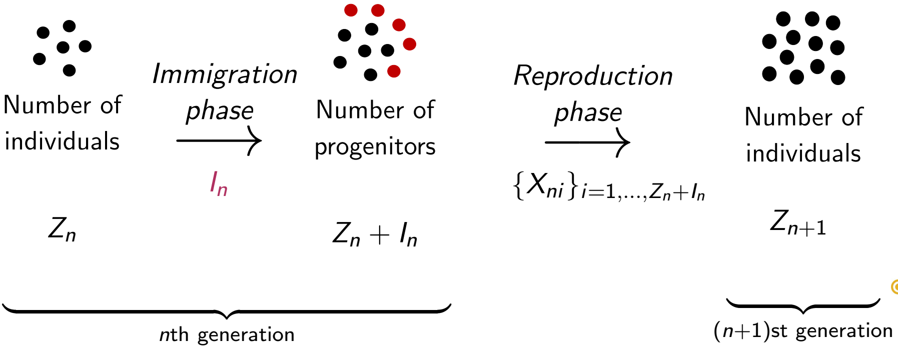

Probability model

- Two independent families of independent $\mathbb{N}_0$-valued r.v.s $\{X_{nj}: \ n \in \mathbb{N}_0; \ j \in \mathbb{N}\}$ and $\{I_n: \ n \in \mathbb{N}_0\}$.
- For each $n \in \mathbb{N}_0$ fixed, X_{nj}, $j \in \mathbb{N}$, have distribution given by the p.g.f. $f_n(s) = \sum_{k=0}^{\infty} f_n[k]s^k$ (offspring distribution of the n-th generation).
- For each $n \in \mathbb{N}_0$, I_n has distribution defined by the p.g.f. $h_n(s) = \sum_{k=0}^{\infty} h_n[k]s^k$, with $h_n(0) < 1$, for $n \in \mathbb{N}_0$ (immigration law of the n-th generation).

The process $\{Z_n\}_{n\in\mathbb{N}_0}$ defined recursively as

$$Z_0 = 0, \quad Z_{n+1} = \sum_{j=1}^{Z_n+I_n} X_{nj}, \quad n \in \mathbb{N}_0,$$

is called *branching process in varying environment $v = \{f_0, f_1, f_2, \ldots\}$ with generation-dependent immigration (BPVEI)* and initial value $Z_0 = 0$.

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Probability model

- **Offspring distribution**:

$$f_{i,n} = f_{i+1} \circ \ldots \circ f_n, \quad -1, \ldots, n$$
$$m_n = E[X_{n1}],$$
$$\sigma_n^2 = Var[X_{n1}].$$

- **Immigration law**:

$$\alpha_n = E[I_n],$$
$$\beta_n^2 = Var[I_n],$$

Proposition

For $n \in \mathbb{N}_0$,

$$E[s^{Z_{n+1}}] = \prod_{i=0}^{n} h_i(f_{i-1,n}(s)), \quad s \in [0,1].$$

$$E[Z_{n+1}] = \sum_{i=0}^{n} \alpha_{n-i} \prod_{j=0}^{i} m_{n-j}.$$

$$Var[Z_{n+1}] = \sum_{i=0}^{n} \beta_i^2 \prod_{j=i}^{n} m_j^2 + \sum_{i=0}^{n} \prod_{j=i+1}^{n} m_j^2 \sigma_i^2 \left(\alpha_i + \sum_{k=0}^{i} \alpha_{i-k} \prod_{l=0}^{k} m_{i-l} \right).$$

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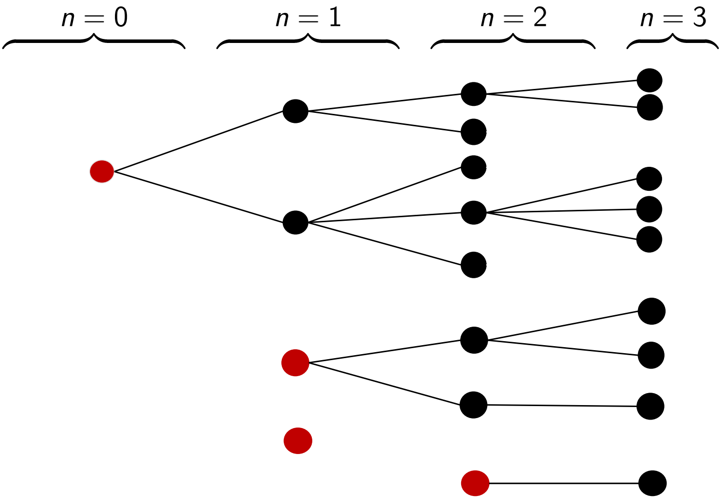
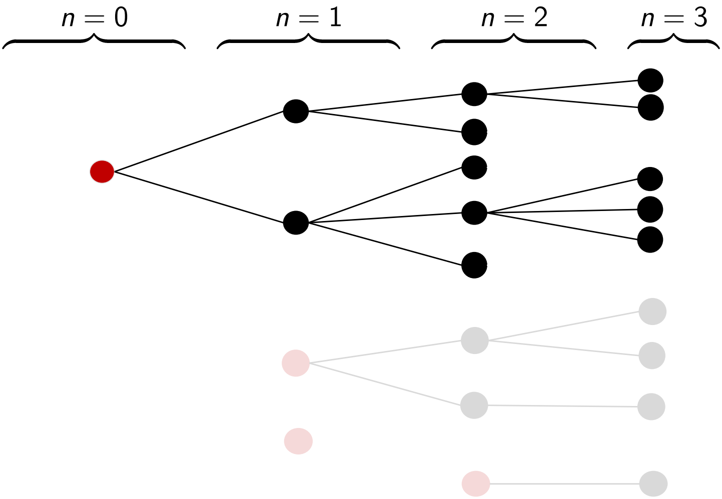


$Z_0 = 0, \quad I_0 = 1, \quad Z_1 = 2, \quad I_1 = 2, \quad Z_2 = 7, \quad I_2 = 1, \quad Z_3 = 10.$



$\{\widetilde{Z}_k^{(0)}\}_{k\in\mathbb{N}_0}$

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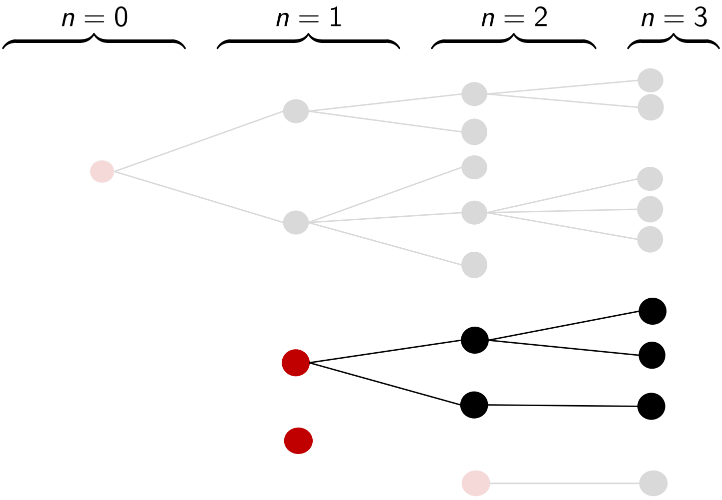

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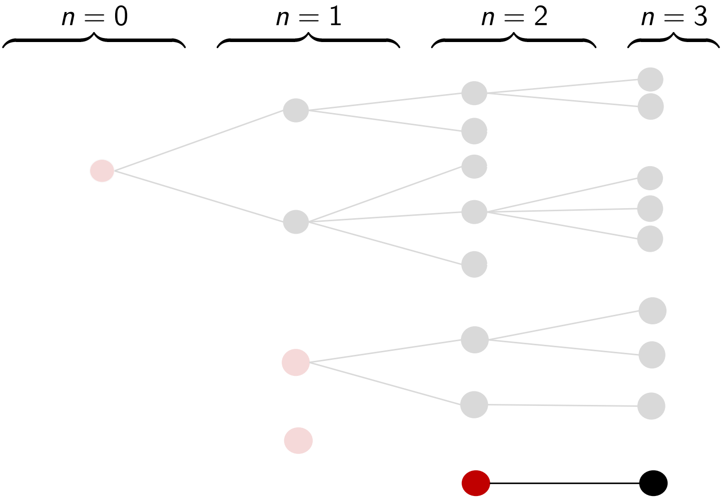

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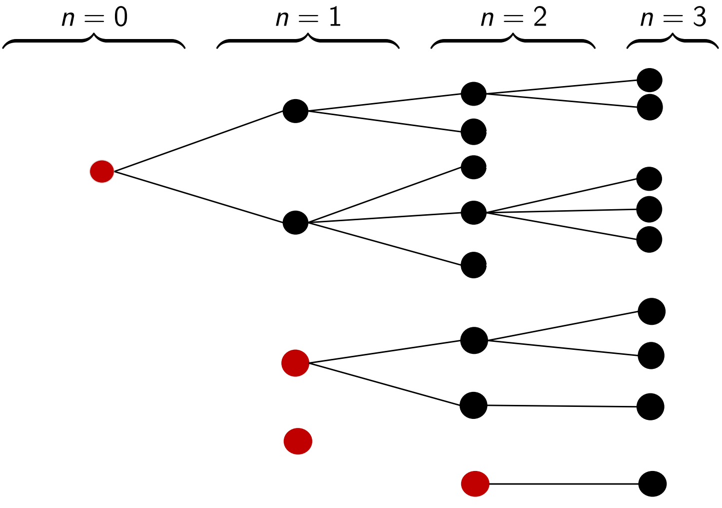

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$\{\widetilde{Z}^{(1)}_k\}_{k\in\mathbb{N}_0}$

$\{\widetilde{Z}^{(2)}_k\}_{k\in\mathbb{N}_0}$

$$Z_3 \stackrel{d}{=} \tilde{Z}^{(0)}_3 + \tilde{Z}^{(1)}_2 + \tilde{Z}^{(2)}_1$$

Proposition

Let us consider:

- *A family of independent r.v.s* $\{X_{ki}^{(j)} : k \in \mathbb{N}_0; i \in \mathbb{N}; j \in \mathbb{N}_0\}$ *such that for each* $k \in \mathbb{N}_0$ *and* $j \in \mathbb{N}_0$ *fixed,* $X_{ki}^{(j)}$*,* $i \in \mathbb{N}$ *are distributed according to the p.g.f.* f_{k+j}*.*
- *The independent processes* $\{\widetilde{Z}_k^{(j)}\}_{k\in\mathbb{N}_0}$*,* $j \in \mathbb{N}_0$*, defined as:*

$$\widetilde{Z}_0^{(j)} = I_j, \quad \widetilde{Z}_{k+1}^{(j)} = \sum_{i=1}^{\widetilde{Z}_k^{(j)}} X_{ki}^{(j)}, \quad k \in \mathbb{N}_0.$$

Then

$$Z_n \stackrel{d}{=} \sum_{j=0}^{n-1} \widetilde{Z}_{n-j}^{(j)}, \quad n \in \mathbb{N}.$$

Extinction problem

- But **0 is not an absorbing state**!

$$P[Z_{n+1} > 0 | Z_n = 0] = 1 - h_n(f_n(0)) > 0$$

... so why is the extinction problem interesting?

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An example

$$P[I_0 = 0] = P[I_1 = 0] = \frac{1}{2}, \quad \text{and} \quad P[I_0 = 1] = P[I_1 = 1] = \frac{1}{2},$$

$$P[I_n = 0] = 1 - \frac{1}{n^2}, \quad \text{and} \quad P[I_n = 1] = \frac{1}{n^2}, \quad n \geq 2,$$

$$X_{n1} \sim \mathcal{P}(m_n), \quad \text{with } m_n = \begin{cases} 1, & n = 0, 1, \\ 1 - \frac{1}{n^2}, & n \geq 2. \end{cases}$$

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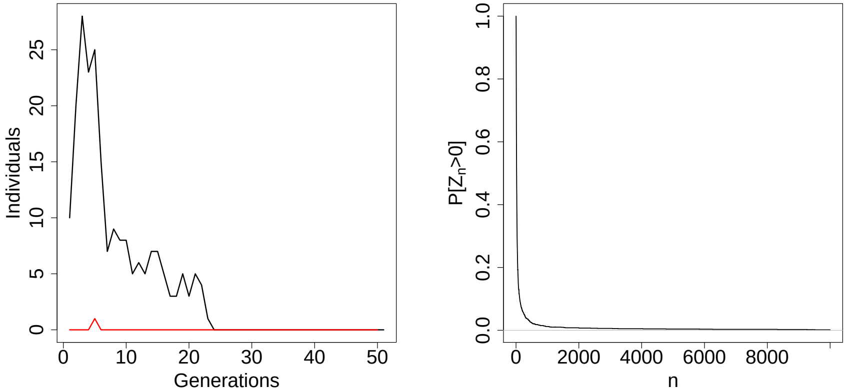


Fig: Left: evolution of the number of individuals (black line) and immigrants (red line). Right: evolution of the probability $P[Z_n > 0]$.

Extinction problem

$$q = P\left[\bigcup_{n=0}^{\infty}\bigcap_{j=n}^{\infty}\{Z_j = 0\}\right]$$

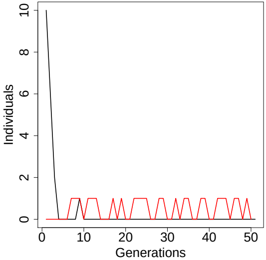
Proposition

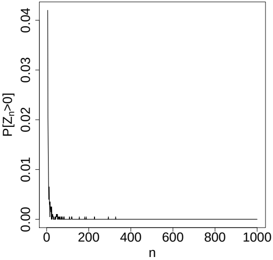
$$q = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} f_{-1,n}(0) = 1, \quad \textit{and} \quad \sum_{j=0}^{\infty}(1 - h_j(f_j(0))) < \infty.$$

Example: $q = 1$

$$P[I_n = 0] = P[I_n = 1] = \frac{1}{2}, \quad n \in \mathbb{N}_0$$

$$1 - P[X_{n1} = 0] = P[X_{n1} = 1] = \begin{cases} 1/2, & n = 0, 1, \\ 1/n^2, & n \geq 2. \end{cases}$$





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Example: $q < 1$

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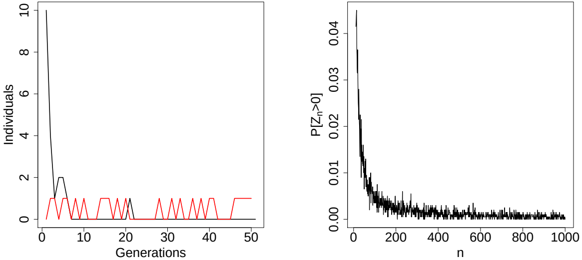

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Asymptotic distribution

- Let us also write

$$\nu_n = \frac{f_n''(1)}{f_n'(1)}, \quad n \in \mathbb{N}_0, \qquad \mu_n = \begin{cases} 1, & n = -1, \\ \prod_{i=0}^{n} m_n, & n \in \mathbb{N}_0. \end{cases}$$

- The regularity assumption (Kersting (2017)): for every $\epsilon > 0$ there is a constant $c_\epsilon < \infty$ such that for all $n \in \mathbb{N}_0$,

$$E[X_{n1}^2; X_{n1} > c_\epsilon(1 + E[X_{n1}])] \leq E[X_{n1}^2; X_{n1} \geq 2]. \qquad (1)$$

- BPVEs with inhomogeneous immigration and critical offspring distributions according to the classification in Kersting (2017) for BPVEs

$$\sum_{k=0}^{n} \frac{\nu_k}{\mu_{k-1}} \to \infty \quad \text{and} \quad \frac{1}{\mu_n} = o\left(\sum_{k=0}^{n} \frac{\nu_k}{\mu_{k-1}}\right), \quad \text{as } n \to \infty. \qquad (2)$$

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Asymptotic distribution

Theorem

Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a BPVEI satisfying (1) and (2) and denote

$$a_{n+1} = \frac{\mu_n}{2}\sum_{k=0}^{n}\frac{\nu_k}{\mu_{k-1}}, \quad n \in \mathbb{N}_0.$$

Assume that

- $\nu_n \to \nu > 0$ *and* $\alpha_n \to \alpha > 0$, *as* $n \to \infty$.
- $\inf_{n\in\mathbb{N}_0} h_n(0) > 0$.
- $\sup_{n\in\mathbb{N}_0} h_n''(1) < \infty$.

Then, the asymptotic distribution of Z_n/a_n is a Gamma distribution with parameters $2\alpha/\nu$ and 1.

Proof

An example

Corollary

Let $\{Z_n\}_{n\in\mathbb{N}_0}$ *be a BPVEI satisfying the conditions of the previous Theorem,*

$$P[Z_n > 0] \to 1, \quad \text{as } n \to \infty.$$

Asymptotic distribution: an example

$$h_n(s) = \begin{cases} 2^{-1}(1+s), & n = 0,1,2 \\ 2^{-1} - n^{-1} + (2^{-1} + n^{-1})s^3, & n \geq 3. \end{cases}$$

$$X_{n1} \sim \mathcal{P}(m_n), \quad \text{with } m_n = \begin{cases} 1, & n = 0,1, \\ 1 - \frac{1}{n^2}, & n \geq 2. \end{cases}$$

$$\nu_n = 1, \quad n \in \mathbb{N}_0.$$

$$a_n = \frac{1}{2} \prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) \left(3 + \sum_{k=3}^{n} \prod_{j=0}^{k-1} \left(1 - \frac{1}{n^2}\right)^{-1}\right), \quad n \geq 2.$$

$$\alpha_n = \frac{3}{2} + \frac{3}{n} \to \alpha = \frac{3}{2}, \quad \text{as } n \to \infty$$

Asymptotic distribution: an example

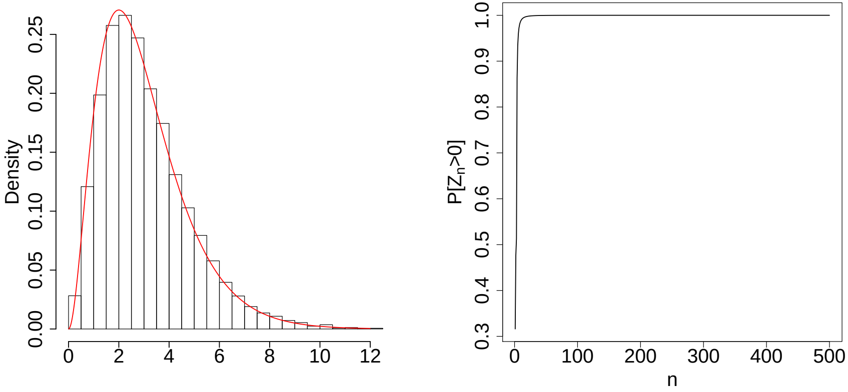

Fig: Left: Comparison of the histogram of Z_n/a_n, for $n = 100$ and a pool of 10^4 BPVEIs, and the density function of the corresponding gamma distribution, which in this case is gamma distribution with parameters 3 and 1 (red line). Right: evolution of the probability $P[Z_n > 0]$.

Conclusions

- For a branching process in varying environment with generation-dependent immigration, we have studied its basic properties such as its **moments** and its **probability generating functions**.
- We have determined a necessary and sufficient condition for the **almost sure extinction** of a branching process in varying environment with time-dependent immigration.
- We have established the **asymptotic distribution** of a branching process in varying environment with time-dependent immigration once suitably normalized.

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References

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Thank you very much!

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Asymptotic distribution: the proof

We make use of the shape functions corresponding to the p.g.f.s of the reproduction laws. The shape function of the p.g.f. f_k, $k \in \mathbb{N}_0$, is the function $\varphi_k : [0,1) \to \mathbb{R}$ satisfying

$$\frac{1}{1-f_k(s)} = \frac{1}{m_k(1-s)} + \varphi_k(s), \quad s \in [0,1).$$

Lemma

Let $i = 0, \ldots, n$ be fixed. Under the assumptions of the Theorem,

$$\sup_{s\in[0,1]} \left| \sum_{k=i}^{n} \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} - \sum_{k=i}^{n} \frac{\varphi_k(1)}{\mu_{k-1}} \right| = o\left(\sum_{k=i}^{n} \frac{\varphi_k(1)}{\mu_{k-1}} \right), \quad \textit{as } n \to \infty.$$

Asymptotic distribution: the proof

Let us fix $\lambda > 0$ and for simplicity, let us denote $s_n = e^{-\lambda/a_n}$, $n \in \mathbb{N}$.

$$E\left[e^{-Z_{n+1}\lambda/a_{n+1}}\right] = \exp\left\{ -\sum_{i=0}^{n} \alpha_i (1 - f_{i-1,n}(s_{n+1})) + \frac{1}{2}\sum_{i=0}^{n}\left(\frac{h_i''(\xi_{in})}{h_i(\xi_{in})} - \frac{h_i'(\xi_{in})^2}{h_i(\xi_{in})^2}\right)(1 - f_{i-1,n}(s_{n+1}))^2 \right\},$$

with $f_{i-1,n}(s_{n+1}) < \xi_{in} < 1$, $i = 0, \ldots, n$, $n \in \mathbb{N}_0$.
The result yields by proving the following convergences:

$$\sum_{i=0}^{n} \alpha_i (1 - f_{i-1,n}(s_{n+1})) \to \log(1+\lambda)^{\frac{2\alpha}{\nu}},$$

$$\sum_{i=0}^{n}\left(\frac{h_i''(\xi_{in})}{h_i(\xi_{in})} - \frac{h_i'(\xi_{in})^2}{h_i(\xi_{in})^2}\right)(1 - f_{i-1,n}(s_{n+1}))^2 \to 0.$$

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