Some Asymptotic Properties of Branching Particle Systems*

José Alfredo López-Mimbela

Centro de Investigación en Matemáticas,
Apartado postal 402, 36000 Guanajuato, Mexico
jalfredo@cimat.mx

*Based on a joint work with Péter Kevei

The Model

Consider a branching population in $\mathbb{R}^d$ of particles of different types $i \in \mathbf{K} := \{1, \ldots, K\}$.

- Each particle of type i moves according to a symmetric α_i-stable motion.
- Its random lifetime has non-arithmetic distribution function Γ_i.
- At death it branches according to a multitype offspring distribution with generating function $f_i(\mathbf{s})$, $\mathbf{s} \in [0,1]^K$, $i \in \mathbf{K}$.
- The descendants appear where the parent individual died, and evolve independently in the same manner.
- We assume that the motions, lifetimes and branchings of particles are independent.

Previous work

- In critically branching and migrating populations, mobility of individuals counteracts the tendency to asymptotic local extinction caused by the clumping effect of the branching.
- In fact, convergence to a non-trivial equilibrium may occur in a spatially distributed population whose members perform migration and reproduction, even if the branching is critical, provided that the mobility of individuals is strong enough.
- This behavior has been investigated in several branching models, including
  - branching random walks (Kallenberg, Matthes *et al* [2, 3]),
  - monotype Markov branching systems (Gorostiza & Wakolbinger [1]),
  - multitype branching systems (Fleischmann & Vatutin, Gorostiza, Roelly & Wakolbinger, López-Mimbela & Wakolbinger [F&V, GRW, L-M,W]),
  - monotype age-dependent branching systems (Vatutin & Wakolbinger [V&W]).

Other Assumptions

- In addition, we assume that the process starts off at time 0 from a Poisson random population, with a given intensity measure, and that all particles at time 0 have age 0.
- Let $M = (m_{i,j})_{i,j=1}^K$ denote the mean matrix of the multitype branching law, that is
$$m_{i,j} = \frac{\partial f_i}{\partial x_j}(\mathbf{1}),$$
where $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^K$. We assume that
$$\mathbf{f}(\mathbf{s}) = (f_1(\mathbf{s}), \dots, f_K(\mathbf{s})) \neq M\mathbf{s},$$
and that M is an ergodic stochastic matrix. This implies that the branching is critical, i.e. the largest eigenvalue of M is 1.

We prove extinction theorems in the following cases:

1. all the particle lifetimes have finite mean, or
2. there is a type whose lifetime distribution has heavy tail, and the other lifetimes have finite mean.

When all particle lifetimes have finite mean we obtain that the process suffers local extinction if

$$d < \alpha/\beta,$$

where the mobility parameter $\alpha = \min_{1\leq i\leq K} \alpha_i$ is the same as in the Markovian case [L-M&W], and the offspring variability parameter $\beta \in (0,1]$ is determined by

$$x - \langle \mathbf{v}, \mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{u}\, x)\rangle \sim x^{1+\beta} L(x) \text{ as } x \to 0.$$

Here

- $\mathbf{v}$ denotes the (normalized) left eigenvector of the matrix M corresponding to the eigenvalue 1, and
- L is slowly varying at 0 in the sense that $\lim_{x\to 0} L(\lambda x)/L(x) = 1$ for every $\lambda > 0$.

Next we assume that exactly one particle type is long-living, i.e. its lifetime distribution has a power tail decay $t^{-\gamma}$, $\gamma \in (0, 1]$, while the other lifetime types have distributions with tails decaying not slower than $A\, t^{-\eta}$ for some $\eta > 1$, $A > 0$.

We consider two scenarios:

1. We assume that the most mobile particle type is, at the same time, long-living, and we prove that extinction holds when $d < \alpha\gamma/\beta$.
2. The most mobile particle type corresponds to a finite-mean lifetime.

   In this scenario, it turns out that local extinction of the population is determined by a complex interaction of the parameters (offspring variability, mobility, longevity) of the long-living type and those of the most mobile type.

   Assuming without loss of generality that type 1 is the long-living type, we prove that the systems suffers local extinction provided that $d < d_+$, where

$$d_+ = \frac{\gamma}{\frac{(\beta+1)\gamma}{\alpha} - \frac{1}{\alpha_1}}.$$

Proofs of some extinction results

Let N_t denote the particle system at time t, i.e. N_t is the point measure on $\mathbb{R}^d \times \mathbf{K}$ determined by the positions and types of individuals alive at time $t \geq 0$.

Let $\mathbf{h} : \mathbb{R}^d \times \mathbf{K} \to [0, \infty)$ be continuous function with compact support. We write

$$\langle \mu, \mathbf{h} \rangle = \int \mathbf{h} \, \mathrm{d}\mu \text{ for any measure } \mu \text{ on } \mathcal{B}(\mathbb{R}^d \times \mathbf{K}).$$

Without danger of confusion we also write $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{K} x_i y_i$ for the scalar product of vectors $\mathbf{x} = (x_1, \ldots, x_K)$ and $\mathbf{y} = (y_1, \ldots, y_K)$.

Assume that the initial population N_0 is a Poisson process with intensity measure

$$\Lambda = \lambda_1 \, \ell \delta_{\{1\}} + \cdots + \lambda_K \, \ell \delta_{\{K\}},$$

where ℓ is d-dimensional Lebesgue measure, and λ_i, $i \in \mathbf{K}$, are positive constants.

The Laplace transform of our branching process is, for any $t \geq 0$, given by

$$\begin{aligned}\mathbf{E}\left[\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle}\right] &= \exp\left\{-\sum_{j=1}^{K}\lambda_j\int_{\mathbb{R}^d}\mathbf{E}_{x,j}\left[1-\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle}\right]\mathrm{d}x\right\}\\ &= \exp\left\{-\left\langle \Lambda, \mathbf{1}-\mathbf{E}_{\cdot,\cdot}\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle}\right\rangle\right\},\end{aligned}$$

where $\mathbf{1} = (1,\ldots,1)$. We put

$$U_i(\mathbf{h},t,x) = \mathbf{E}_{x,i}\left(1-\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle}\right).$$

The lower index in $\mathbf{P}$ and $\mathbf{E}$ refers to the initial distribution. In particular, $\mathbf{P}_{x,i}$ and $\mathbf{E}_{x,i}$ refer to a population having an ancestor $\delta_{(x,i)}$ of type $i \in \mathbf{K}$, initially at position $x \in \mathbb{R}^d$.

By extinction of $\{N_t,\ t \geq 0\}$ here we mean that the Laplace transform of N_t converges to the Laplace transform of the empty population, and for this it is enough to verify that

$$\langle \Lambda, U_\cdot(\mathbf{h},t,\cdot)\rangle \to 0 \quad \text{as } t \to \infty,$$

which is the same as showing that

$$\int \left[\mathbf{E}_{x,i}\left(1-\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle}\right)\right]\mathrm{d}x \to 0 \quad \text{as } t\to\infty \text{ for all } i\in\mathbf{K}.$$

Let $B\subset\mathbb{R}^d$ be a ball, and assume that $B\times\mathbf{K}\supset \mathrm{supp}\mathbf{h}$. Then

$$1-\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle} \le I(N_t(B\times\mathbf{K})>0),$$

which implies

$$\mathbf{E}_{x,i}\left[1-\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle}\right] \le \mathbf{P}_{i,x}\left\{N_t(B\times\mathbf{K})>0\right\}.$$

Conversely, if $\mathbf{h}|_{B\times\mathbf{K}}\ge 1$, then

$$1-\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle} \ge (1-\mathrm{e}^{-1})I(N_t(B\times\mathbf{K})>0),$$

and so

$$\mathbf{E}_{x,i}\left[1-\mathrm{e}^{-\langle N_t,\mathbf{h}\rangle}\right] \ge (1-\mathrm{e}^{-1})\mathbf{P}_{i,x}\left\{N_t(B\times\mathbf{K})>0\right\}.$$

In this way we get that

Lemma 1 *Extinction of $\{N_t,\, t \geq 0\}$ occurs if, and only if for any bounded Borel set $B \subset \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \mathbf{P}_{i,x}\{N_t(B \times \mathbf{K}) > 0\}\,\mathrm{d}x \to 0 \textit{ for all } i \in \mathbf{K}, \textit{ as } t \to \infty.$$

Lemma 2 (Fleischmann & Vatutin) *Put $\alpha = \min_{i \in \mathbf{K}} \alpha_i$. For each bounded $B \subset \mathbb{R}^d$*

$$\sup_{t \geq 1} \int_{\mathbb{R}^d \setminus C(t,L)} \mathbf{E}_{x,i} N_t(B \times \mathbf{K})\mathrm{d}x \longrightarrow 0 \textit{ as } L \uparrow \infty,$$

where $C(t,L) = \{x \in \mathbb{R}^d : |x| \leq L t^{1/\alpha}\}$.

This means that extinction of $\{N_t,\, t \geq 0\}$ occurs if, and only if for any bounded Borel set $B \subset \mathbb{R}^d$, and for L large enough

$$\int_{C(t,L)} \mathbf{P}_{i,x}\{N_t(B \times \mathbf{K}) > 0\}\,\mathrm{d}x \to 0 \text{ for all } i \in \mathbf{K} \text{ as } t \to \infty. \tag{1}$$

Let $F^{(i)}$ denote the probability generating function of the process starting from a single particle of type i:

$$F^{(i)}(t; s_1, \ldots, s_K) = \mathbf{E}_i \left[s_1^{N_t^1(\mathbb{R}^d)} \cdots s_K^{N_t^K(\mathbb{R}^d)} \right], \quad 0 \le s_j \le 1,\ j \in \mathbf{K}. \quad (2)$$

Put

$$\begin{aligned} Q^{(i)}(t; s_1, \ldots, s_K) &= 1 - F^{(i)}(t; s_1, \ldots, s_K), \\ q^{(i)}(t; s) &= Q^{(i)}(t; s, \ldots, s). \end{aligned}$$

Consider the discrete–time multitype Galton–Watson process $\{\mathbf{X}_n\}$, with the same offspring distributions as in the branching particle system.

Let $\mathbf{v}$ and $\mathbf{u}$ respectively denote the left and right normed eigenvectors of the mean matrix M, which are determined by:

$$\mathbf{v}M = \mathbf{v}, \quad M\mathbf{u} = \mathbf{u}, \quad \mathbf{v}\mathbf{u} = 1, \quad \mathbf{1}\mathbf{u} = 1. \quad (3)$$

Since by assumption M is stochastic, $\mathbf{u} = K^{-1}\mathbf{1}$.

Let $\mathbf{f}_n = (f_n^1, \ldots, f_n^K)$ be the generating function of the n^{th} generation, i.e $f_n^i(\mathbf{x}) = \mathbf{E}_i \left[\mathbf{x}^{\mathbf{X}_n} \right]$ and put $\mathbf{f}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x})$. Then $\mathbf{f}_{n+1}(x) = \mathbf{f}(\mathbf{f}_n(\mathbf{x}))$.

Let us assume that

$$x - \langle \mathbf{v}, \mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{u}\, x)\rangle \sim x^{1+\beta} L(x) \quad \text{as } x \to 0, \tag{4}$$

where $\beta \in (0, 1]$ and L is slowly varying at 0 in the sense that $\lim_{x\to 0} L(\lambda x)/L(x) = 1$ for every $\lambda > 0$. In this case, for the survival probabilities it is known that

$$\mathbf{1} - \mathbf{f}_n(0) = (\mathbf{u} + o(1))n^{-1/\beta} L_1(n) \text{ as } n \to \infty, \tag{5}$$

where L_1 is slowly varying at ∞ (see Theorem 1 in [Vat77] or Theorem 1 in [Vat78]). Moreover, assume that

$$\lim_{n\to\infty} \frac{n[1 - \Gamma_i(n)]}{\langle \mathbf{v}, \mathbf{1} - \mathbf{f}_n(0)\rangle} = 0, \quad i = 1, 2, \ldots, K. \tag{6}$$

Then

$$Q^{(i)}(t; 0) = \mathbf{P}_i \{\text{the process is not extinct at } t\} \sim u_i\, D^{\frac{1}{\beta}}\, t^{-\frac{1}{\beta}} L_1(t) \text{ as } t \to \infty, \tag{7}$$

where $D = \sum_{i=1}^{K} u_i v_i \mu_i$; see Theorem 2 in [Vat78].

Using the estimate above, we obtain the following theorem.

Theorem 3 *Assume that* (4) *and* (6) *hold. Then for* $d < \alpha/\beta$ *the process* $\{N_t,\ t \geq 0\}$ *suffers local extinction.*

Proof. Due to (7), for any $\varepsilon > 0$

$$Q^{(i)}(t;0) \leq c\, t^{-\frac{1-\varepsilon}{\beta}}, \quad i \in \mathbf{K}.$$

Clearly

$$\mathbf{P}_{i,x}\left\{N_t(B \times \mathbf{K}) > 0\right\} \leq \mathbf{P}_i\left\{\text{the process is not extinct at time } t\right\}.$$

Plugging this into (1) we get

$$\int_{C(t,L)} \mathbf{P}_{i,x}\left\{N_t(B \times \mathbf{K}) > 0\right\} \mathrm{d}x \leq c\, t^{\frac{d}{\alpha} - \frac{1-\varepsilon}{\beta}}.$$

Since by assumption $d < \alpha/\beta$, for some $\varepsilon > 0$ the exponent of t in the above inequality is negative, which implies that the integral in the left-hand side tends to 0.

Remark. When the generating functions f_i, $i \in \mathbf{K}$, are of the form $f_i(s, \ldots, s) = f_i(s) = s + c\,(1-s)^{1+\beta_i}$ where $\beta_i \in (0,1]$, it is easy to verify that (4) holds with $\beta = \min\{\beta_i : i \in \mathbf{K}\}$, and that (6) is fulfilled if for some $\varepsilon > 0$

$$\lim_{n\to\infty} n^{1+\frac{1}{\beta}+\varepsilon}[1 - \Gamma_i(n)] = 0, \quad i = 1, 2, \ldots, K.$$

A lifetime with infinite mean – Case A

From now on assume that there is exactly one lifetime distribution with infinite mean; more precisely we let $\gamma \in (0, 1]$ and assume that

$$1 - \Gamma_1(x) \sim x^{-\gamma}, \quad \text{as } x \to \infty \text{ and} \qquad (8)$$
$$1 - \Gamma_j(x) \leq A x^{-\eta_j}, \quad j = 2, 3, \ldots, K,$$

where $A > 0$ and $\eta_j > 1$, $j = 2, 3, \ldots, K$.
Put

$$\eta = \min_{j=2,3,\ldots,K} \eta_j.$$

Moreover, in this part we additionally assume that

$$\alpha = \min_{i \in \mathbf{K}} \alpha_i = \alpha_1,$$

that is, the long-living particle type is the most mobile as well.

A key tool is an analogue of Lemma 3 in [V&W]. Recall the notations after (2). The proof is a multidimensional extension of the proof in [V&W].

Lemma 4 *If $\eta - 1 > d/\alpha$, there exists a constant $c_2 > 0$ such that for any $x \in \mathbb{R}^d, t > 0, i \in \mathbf{K}$ and $u \in (0, t - c_2^{\alpha/d})$,*

$$\mathbf{P}_{x,i}\left\{N_t(B \times \mathbf{K}) > 0\right\} \leq q^{(i)}\left(u; 1 - c_2\left(t-u\right)^{-d/\alpha}\right).$$

Let us define the set

$$\Lambda = \{\mathbf{s} \in [0,1]^K : \mathbf{f}(\mathbf{s}) \geq \mathbf{s}\}, \tag{9}$$

where an inequality of the form $(x_1, \dots, x_K) \geq (y_1, \dots, y_K)$ means here that $x_i \geq y_i$ for $i = 1, 2, \dots, K$.

We remark that, since

$$\mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{u}x) \leq M\mathbf{u}x = \mathbf{u}x,$$

we have $\mathbf{1} - \mathbf{u}x \in \Lambda$ for all x with $0 < \mathbf{u}x \leq \mathbf{1}$. In our case $\mathbf{u} = K^{-1}\mathbf{1}$, and this implies that Λ contains the diagonal $\{(s, \dots, s) : s \in [0, 1]\}$.

For given matrix families $A(t) = (a_{ij}(t))_{i,j}$ and $B(t) = (b_{ij}(t))_{i,j}$, $t \geq 0$, let us define the matrix convolution by

$$C = A * B = (c_{ij}(t))_{i,j} \text{ with } c_{ij}(t) = \sum_{k=1}^{K} \int_0^t a_{ik}(t-s) b_{kj}(\mathrm{d}s).$$

The convolution of a matrix and a vector is defined similarly. Put $M_\Gamma^1(t) = (m_{ij}\Gamma_i(t))_{i,j}$ and recursively define

$$M_\Gamma^{n+1}(t) = M_\Gamma^1(t) * M_\Gamma^n(t), \quad n = 1, 2, \dots.$$

Put also $M_\Gamma^0(t) = (\delta_{ij}\Gamma_i^0(t))_{i,j}$, where $\Gamma_i^0(t)$ is the distribution function of a constant 0 random variable.

Notice that $M_\Gamma^0(t)$ constitutes the unit element in matrix convolution. The following multidimensional comparison lemma is borrowed from [Vatutin '78]:

Comparison Lemma 1. *Let* $\Gamma = (\Gamma_1, \ldots, \Gamma_K)$*. For any* $t > 0$*, any natural* n *and for all* $\mathbf{s} \in \Lambda$*,*

$$\begin{aligned} \mathbf{1} - \mathbf{f}_n(\mathbf{s}) - M_\Gamma^n * [(\mathbf{1} - \mathbf{s}) \otimes \Gamma](t) \\ \leq \quad & \mathbf{1} - F(t; \mathbf{s}) \\ \leq \quad & \mathbf{1} - \mathbf{f}_n(\mathbf{s}) + \sum_{j=0}^{n-1} M_\Gamma^j * [(\mathbf{1} - \mathbf{s}) \otimes [\mathbf{1} - \Gamma]](t). \end{aligned}$$

Here $\mathbf{x} \otimes \mathbf{y} := (x_1 y_1, x_2 y_2, \ldots, x_K y_K)$ if $\mathbf{x} = (x_1, \ldots, x_K)$ and $\mathbf{y} = (y_1, \ldots, y_K)$.

We are also going to use the following comparison result [Vatutin '79]:

Comparison Lemma 2. *Consider two critical multitype branching processes sharing the same branching mechanism, with corresponding lifetime distributions*

$$\Gamma(t) = (\Gamma_1(t), \ldots, \Gamma_K(t)) \text{ and } \Gamma^*(t) = (\Gamma_1^*(t), \ldots, \Gamma_K^*(t)).$$

Assume that $\Gamma(t) \geq \Gamma^*(t)$ *for all* $t \geq 0$. *Then for all* $t \geq 0$ *and* $\mathbf{s} \in \Lambda$,

$$F(t; \mathbf{s}) \geq F^*(t; \mathbf{s}),$$

where F *and* F^* *are, respectively, the vector generating functions of the number of particles at time* t *in the first and second process.*

Theorem 5. *Assume that*

$$x - \langle \mathbf{v}, \mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{u}\, x)\rangle \sim x^{1+\beta} L(x) \quad as\ x \to 0,$$

the mean matrix M is stochastic, and the lifetimes satisfy

$1 - \Gamma_1(t) \sim t^{-\gamma}$ *for some* $\gamma \le 1$*, and* $1 - \Gamma_j(x) \le A\, x^{-\eta_j}$, $\quad j = 2, 3, \ldots, K$,

where $\eta_j > 1$*,* $j = 2, 3, \ldots, K$*. Put* $\eta = \min\{\eta_j : j = 2, 3, \ldots, K\}$*. If* $\eta - 1 > d/\alpha$ *and* $d < \frac{\alpha\gamma}{\beta}$*, then the process suffers local extinction.*

Proof. Define the distribution function

$$\tilde{\Gamma}(t) = \prod_{i=1}^{K} \Gamma_i(t),$$

which is the distribution function of $\tilde{\xi} = \max\{\xi_1, \ldots, \xi_K\}$, where the random variables ξ_i, $i = 1, \ldots, K$, are independent with distribution function Γ_i. Lemma 6 below shows that, asymptotically,

$$1 - \tilde{\Gamma}(t) \sim t^{-\gamma}.$$

Moreover,

$$(\tilde{\Gamma}(t), \ldots, \tilde{\Gamma}(t)) \le (\Gamma_1(t), \ldots, \Gamma_K(t)).$$

Consider a new branching process where the branching mechanism is unchanged, but the lifetimes of all types have distribution $\tilde{\Gamma}$. Let $\tilde{F}(t;\mathbf{s})$ denote its generating function at time t.

Clearly, the choice of $\tilde{\Gamma}$ shows that **Comparison Lemma 2** is applicable, and so for $\mathbf{s} \in \Lambda$,

$$\tilde{F}(t;\mathbf{s}) \leq F(t;\mathbf{s}). \tag{10}$$

(Notice that Λ, as defined in (9), depends only on the branching mechanism of our process). Now we apply the **Comparison Lemma 1** for this new process. Since now all the lifetimes have the same distribution,

$$M^n_{\tilde{\Gamma}}(t) = M^n\, \tilde{\Gamma}^{*n}(t),$$

where ^{*n} stands for the n-fold convolution. Moreover, for $\mathbf{s} = s\,\mathbf{1}$,

$$\begin{aligned} & M^j_{\tilde{\Gamma}} * [(\mathbf{1}-\mathbf{s})(\mathbf{1}-\tilde{\Gamma})](t) \\ = \;& (1-s)(\tilde{\Gamma}^{*j}(t) - \tilde{\Gamma}^{*(j+1)}(t))M^j\mathbf{1} = (1-s)(\tilde{\Gamma}^{*j}(t) - \tilde{\Gamma}^{*(j+1)}(t))\mathbf{1}, \end{aligned}$$

where we used the simple fact that M^j is stochastic if M is stochastic.

Thus, in the rightmost inequality of the **Comparison Lemma 1** we get a telescopic sum, and therefore we obtain

$$\mathbf{1} - \tilde{F}(t; \mathbf{1}s) = \tilde{Q}(t; \mathbf{1}s) \leq \mathbf{1} - f_n(\mathbf{1}s) + (1-s)[1 - \tilde{\Gamma}^{*n}(t)]\mathbf{1}.$$

According to (5), for the survival probabilities we have

$$1 - f_n^{(i)}(\mathbf{1}s) \leq 1 - f_n^{(i)}(0) \leq c\, n^{-\frac{1}{\beta}}.$$

Since $\tilde{F}(t; \mathbf{s}) \leq F(t; \mathbf{s})$, we have, for $s \in (0, 1)$,

$$Q^{(i)}(t; \mathbf{1}s) = 1 - F^{(i)}(t; \mathbf{1}s) \leq c\, n^{-\frac{1}{\beta}} + (1-s)\mathbf{P}\left\{S_n > t\right\},$$

where $S_n = \xi_1^{\tilde{\Gamma}} + \cdots + \xi_n^{\tilde{\Gamma}}$ and $\{\xi_i^{\tilde{\Gamma}}\}$ are independent $\tilde{\Gamma}$-distributed r.v.
It can be proved that for any $\varepsilon > 0$ and n large

$$\mathbf{P}\left\{S_n > n^{\frac{1+\varepsilon}{\gamma}}\right\} \leq 2\, n^{-\varepsilon},$$

hence, choosing $n = t^{\gamma/(1+\varepsilon)}$ gives

$$\mathbf{P}\left\{S_n > t\right\} = \mathbf{P}\left\{S_n > n^{\frac{1+\varepsilon}{\gamma}}\right\} \leq 2\, n^{-\varepsilon} = 2\, t^{-\gamma\varepsilon/(1+\varepsilon)}$$

and

$$q^{(i)}(t; 1-s) = Q^{(i)}(t; (1-s)\mathbf{1}) \leq c\, t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + s\, t^{-\gamma\varepsilon/(1+\varepsilon)}.$$

Setting $u = t/2$ in Lemma 4 we obtain the inequality

$$q^{(i)}(u; 1 - c_2\,(t-u)^{-d/\alpha}) \leq c\,t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + c\,t^{-\frac{d}{\alpha}}\,t^{-\frac{\gamma\,\varepsilon}{1+\varepsilon}},$$

moreover,

$$\begin{aligned} \int_{C(t,L)} \mathbf{P}_{i,x}\{N_t(B \times \mathbf{k}) > 0\}\,dx &\leq \int_{C(t,L)} \left(c\,t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + c\,t^{-\frac{d}{\alpha}}\,t^{-\frac{\gamma\,\varepsilon}{1+\varepsilon}}\right) dx \\ &\leq \text{Const.} \left(t^{\frac{d}{\alpha} - \frac{\gamma}{(1+\varepsilon)\beta}} + t^{-\frac{\gamma\,\varepsilon}{1+\varepsilon}}\right) \qquad (11) \\ &\to 0 \text{ as } t \to \infty \end{aligned}$$

because the second term in the right of (11) goes to 0, while in the first one the exponent

$$\frac{d}{\alpha} - \frac{\gamma}{(1+\varepsilon)\beta}$$

is negative if $d < \alpha\gamma/\beta$ and ε is small enough.

The simple lemma we used above is the following:

Lemma 6. *Let* X, Y *be independent non-negative random variables with corresponding distribution functions* F *and* G. *Assume that* $1 - F(x) \sim x^{-\gamma}$ *and* $\mathbf{E}Y < \infty$. *Then for the distribution of* $Z = \max\{X, Y\}$ *we have*

$$1 - H(z) := \mathbf{P}\{Z > z\} \sim z^{-\gamma}, \quad \text{as} \quad z \to \infty.$$

A lifetime with infinite mean – Case B

Now let us investigate the case when α_1 is not the minimal $\alpha = \min\{\alpha_i : i = 1, 2, \ldots, K\}$.

Without loss of generality, let us assume that $\alpha = \alpha_2$.

Notice that Lemma 4 is true in this case with exponent $-d/\alpha_1$. We state it for the easier reference.

Lemma 7. *If $\eta - 1 > d/\alpha_1$, there exists a constant $c_2 > 0$ such for any $x \in \mathbb{R}^d, t > 0, i \in \mathbf{K}$ and $u \in (0, t - c_2^{\alpha_1/d})$,*

$$\mathbf{P}_{x,i}\left\{N_t(B \times \mathbf{K}) > 0\right\} \leq q^{(i)}\left(u; 1 - c_2\,(t-u)^{-d/\alpha_1}\right).$$

Put

$$v = \max\left\{\frac{1}{\alpha_1}, \frac{\gamma}{\alpha}\right\}. \tag{12}$$

Lemma 8. *Assume that $\gamma\eta > d/\alpha + 1$. If $\gamma < 1$, then for any $\varepsilon > 0$, for any $i \in \{1, 2, \ldots, K\}$ and for any bounded Borel set B,*

$$\lim_{t\to\infty} \int_{|x|\geq t^{v+\varepsilon}} \mathbf{P}_{x,i}\left\{N_t(B\times\mathbf{K}) > 0\right\} \mathrm{d}x = 0.$$

For $\gamma = 1$ (then necessarily $v = 1/\alpha$),

$$\lim_{L\to\infty} \limsup_{t\to\infty} \int_{|x|\geq Lt^{v}} \mathbf{P}_{x,i}\left\{N_t(B\times\mathbf{K}) > 0\right\} \mathrm{d}x = 0.$$

The value $\alpha_1\gamma$ can be considered as the *effective mobility* of the type-1 particles.

At an intuitive level if $\alpha_1\gamma > \alpha$, then second particle type is more mobile, even considering the long-living effect of the first one, so that in this case the "dominant" mobility is associated to the second particle type.

The next two theorems deal with the cases when the first type is the dominant and when the second one, respectively.

Theorem. *Assume that* (4) *holds and that* $\gamma\eta > d/\alpha + 1$*. If* $\alpha \geq \alpha_1\gamma$*, i.e. the mobility of the first particle type is dominant, then the process suffers local extinction for* $d < \alpha_1\gamma/\beta$*.*

Proof. Writing $u = t/2$ in Lemma 7, and proceeding in the same way as we did before we get

$$q^{(i)}(t/2; 1 - c\,t^{-d/\alpha_1}) \leq c\,t^{-d/\alpha_1} t^{-\gamma\varepsilon/(1+\varepsilon)} + c\,t^{-\gamma/(1+\varepsilon)\beta}.$$

Since in this case $v = 1/\alpha_1$, from Lemma 8 we get extinction provided that

$$\frac{d}{\alpha_1} < \frac{\gamma}{(1+\varepsilon)\beta},$$

which holds for ε small enough if $d < \alpha_1\gamma/\beta$. .

Theorem. *Assume that* $\gamma\eta > d/\alpha + 1$. *If* $\alpha_1\gamma > \alpha$, *i.e. the mobility of the second particle type is the dominant one, then the process suffers local extinction for* $d < d_+$, *where*

$$d_+ = \frac{\gamma}{\frac{(\beta+1)\gamma}{\alpha} - \frac{1}{\alpha_1}}. \tag{13}$$

Proof. From the **Comparison Lemma 1** we have

$$Q^{(i)}(t; \mathbf{1}s) \leq c\, n^{-\frac{1}{\beta}} + (1-s)\mathbf{P}\left\{S_n \geq t\right\}.$$

We have to choose $t = n^{\frac{1+\varepsilon}{\gamma}}$ for some $\varepsilon > 0$, and then minimize the estimations in ε. In this case

$$q^{(i)}(t; 1-s) \leq c\, t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + s\, t^{-\frac{\varepsilon\gamma}{1+\varepsilon}}.$$

Putting $u = t/2$ in Lemma 7 renders

$$q^{(i)}(t/2; 1 - c_2\, t^{-d/\alpha_1}) \leq c\, t^{-\frac{\gamma}{(1+\varepsilon)\beta}} + c\, t^{-d/\alpha_1 - \frac{\varepsilon\gamma}{1+\varepsilon}}.$$

Therefore we have to maximize

$$\min\left\{\frac{\gamma}{(1+\varepsilon)\beta},\frac{d}{\alpha_1}+\frac{\varepsilon\gamma}{1+\varepsilon}\right\}$$

with respect to ε. Since the term $\gamma/((1+\varepsilon)\beta)$ is monotone decreasing, and the term $d/\alpha_1+\varepsilon\gamma/(1+\varepsilon)$ is increasing in ε, easy computations show that the optimal choice is

$$\varepsilon=\frac{\gamma(1+\beta^{-1})}{d/\alpha_1+\gamma}-1,$$

and the estimation is

$$q^{(i)}(t/2;1-c_2\,t^{-d/\alpha_1})\leq c\,t^{-\frac{d/\alpha_1+\gamma}{1+\beta}}.$$

Combining this with Lemma 8, and taking into account that $v=\gamma/\alpha$, we get extinction if

$$d\frac{\gamma}{\alpha}<\frac{d/\alpha_1+\gamma}{1+\beta}.$$

Solving the inequality, gives that extinction holds for $d<d_+$, with the given dimension d_+.

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