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Diffusion Approximation of Branching Processes in Random Environment

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Introduction

We present here :

1. **Diffusion approximation of near critical Markov branching processes** (BGW, CTMBP). Especially,

- Feller-Jiřina theorem Feller 1951,Jiřina 1969
- Jagers theorem (1971) are revisited

2. **Random Environment Continuous Time Markov Branching processes**

- Averaging
- Diffusion Approximation

by random evolution method.

Some Literature on Diffusion Approximation

Diffusion approximation

- Feller (1951), Jiřina (1969) : DA BGW
- Jagers (1971) : DA CTMBP
- Aliev & Shurenkov (1983) : DA BGW
- Wei & Winnicki (1989) : DA with immigration

... in Random Environment:

- Keiding (1975): conjecture
- Kurtz (1978): established
- Böinghoff & Hutzenthaler (2013): asympt. survival proba
- Dyakonova (2014): REBP

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Bienaymé-Galton-Watson BP (BGW)

Consider a BGW branching process in discrete-time

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{n-1,j}, \quad n \geq 1, \quad Z_0 = 1. \tag{1}$$

Denote by :
$\mu := \mathbf{E}\xi_{n-1,j}$ and
$\sigma^2 := \mathrm{Var}\,(\xi_{n-1,j})$,
the common mean and variance of offspring, $\xi_{n-1,j}$, which are i.i.d. random variables.

BGW process in series scheme

Define now the family of processes in series scheme, indexed by the series parameter $\varepsilon > 0$, say Z_n^ε, and define the processes

$$Y_t^\varepsilon := \varepsilon Z_{[t/\varepsilon]}^\varepsilon$$

for $t \geq 0$, $\varepsilon > 0$. Let P_ε be the transition operator of the MC Z_n^ε, and the discrete operator $Q^\varepsilon = P_\varepsilon - I$.

Near Critical Case: Assumptions

C1: Offspring mean assumption: $\mu_\varepsilon = 1 + \varepsilon\alpha + o(\varepsilon)$, as $\varepsilon \downarrow 0$, and $\alpha \in \mathbb{R}$ a constant;

C2: Offspring variance assumption: $\sigma_\varepsilon^2 = \sigma^2 + o_\varepsilon(1)$, with $0 < \sigma^2 < \infty$.

C3: Initial value assumption: Y_0^ε converge to a point $x \in \mathbf{R}$, as $\varepsilon \downarrow 0$.

Feller-Jiřina Theorem

Theorem

Let Conditions C2-C3 hold. Then Condition C1 is necessary and sufficient that the following weak convergence holds

$$Y_t^\varepsilon \Longrightarrow Y_t, \qquad as \quad \varepsilon \downarrow 0,$$

where Y_t is a diffusion process defined by the generator

$$L\varphi(x) = \alpha x \varphi'(x) + \frac{1}{2}\sigma^2 x \varphi''(x)$$

with initial value $Y_0 = x$, and $\varphi \in C_0^2(\mathbf{R})$. Here φ' and φ'' are the first and second derivative of the function φ.

Markov Branching Process – Continuous-Time

Consider a CT Markov branching process

$$Z_{t+s} = \sum_{i=1}^{Z_t} \xi_i^{(s)} \tag{2}$$

where $\xi_i^{(s)}$ is the number of offspring of the i-th particle living in time t. The particle lifetime follows an exponential distribution with mean $1/\lambda$, $\lambda > 0$.

Jagers Theorem

Let us consider the family of processes $Y_t^\varepsilon := \varepsilon Z_{t/\varepsilon}^\varepsilon$, $t \geq 0$, $\varepsilon > 0$. Then we have the following result.

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Theorem

Let Conditions C2-C3 hold. Then Condition C1 is necessary and sufficient that the following weak convergence holds

$$Y_t^\varepsilon \Longrightarrow Y_t, \qquad \text{as} \quad \varepsilon \downarrow 0,$$

where Y_t is a diffusion process defined by the generator

$$L\varphi(x) = \alpha\lambda x\varphi'(x) + \frac{1}{2}\lambda\sigma^2 x\varphi''(x)$$

with initial value $Y_0 = x$, and $\varphi \in C_0^2(\mathbf{R})$.

Some Remarks

The following cases which provide also averaging results in both discrete and continuous-time could be of some interest.

- For any $a \in \mathbb{R}^*$ and $\alpha \neq 0$, the above diffusions (Th1 and Th2) are transient. In the case where $\alpha = 0$ the Wiener process is transient too.
- Let $\mu_\varepsilon = \mu + \varepsilon^k \alpha + o(\varepsilon^k)$. Then the drift of the limit process is no null iff $\alpha \in \mathbb{R}^*$ and $k = 1$. If $\alpha = 0$ and $k \geq 2$, then the drift is null.
- With $\mu \neq 1$, and $\lambda_\varepsilon = \varepsilon\lambda$, then the limit process is the deterministic function $y(t) = xe^{\lambda(\mu-1)t}$.
- If in condition C2 becomes $\sigma_\varepsilon^2 = o_\varepsilon(1)$, then we get, for both theorems, as limit process, Y_t, the deterministic function $y(t) := xe^{bt}$, with $b = \alpha$ (Th1), and $b = \lambda\alpha$ (Th2).

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CTMBP in Markov Random Environment

We present here two kind of results:

- Average approximation
  BPRE $\Rightarrow$ BP
- Diffusion approximation
  BPRE $\Rightarrow$ DIFUSSION

Random environment

Let us consider a jump Markov process $X_t, t \geq 0$, with state space $(E, \mathcal{E})$, generating operator Q, i.e.,

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)] \tag{3}$$

where q is the intensity of jumps function, a measurable and bounded function on compact sets.

Assumptions

R1: The Markov process X is uniformly ergodic, with ergodic probability $\pi(B), B \in \mathcal{E}$.

R2: In the above series scheme, for any fixed $x \in E$, the r.v. $\xi_j^\varepsilon(x)$, $j \geq 1$, are i.i.d. and are independent on $\{X_t = x\}$.

R3: The law of the number of offspring depends on $x \in E$, i.e., $p_j(x) := \mathbb{P}(\xi_j^\varepsilon(x) = j) = \mathbb{P}(\xi_j^\varepsilon(x) = j \mid X_t = x)$, $x \in E$.

R4: The mean value of $\xi_j^\varepsilon(x)$, has the following representation:
$\mu_\varepsilon(x) := 1 + \varepsilon\mu(x) + o_x(\varepsilon)$;
and the variance:
$\sigma_\varepsilon^2(x) := \sigma^2(x) + o_x(\varepsilon)$, $x \in E$.

R5: The lifetime of the particles are exponentially distributed with parameter depending on the state x of the X_t, $\lambda(x) > 0$, $x \in E$, and such that $\int_E \pi(dx)\lambda(x) < \infty$.

Averaging

Here we consider the coupled process $(Z_t^\varepsilon, X_{t/\varepsilon})$

Theorem

Under assumptions R1-R5, the following weak convergence holds

$$Z_t^\varepsilon \Longrightarrow Z_t^0, \qquad as \quad \varepsilon \downarrow 0,$$

where Z_t^0 is a Markov branching process with offspring probability law $\hat{p}_j, j \geq 0$, and lifetime distibution an exponential one with parameter $\hat{\lambda}$, where

$$\hat{\lambda} := \int_E \pi(dx)\lambda(x), \qquad and \qquad \hat{p}_j := \int_E \pi(dx)\lambda(x)p_j(x)/\hat{\lambda}.$$

Diffusion Approximation

Here we consider the coupled process $(Y_t^\varepsilon = \varepsilon Z^\varepsilon_{t/\varepsilon}, X_{t/\varepsilon})$

Theorem

Under assumptions R1-R5, the following weak convergence holds

$$Y_t^\varepsilon \Longrightarrow Y_t^0, \qquad as \quad \varepsilon \downarrow 0,$$

where Y_t^0 is a diffusion process defined by the generator

$$L\varphi(z) = \hat{a} z \varphi'(z) + \frac{1}{2}\hat{b}^2 z \varphi''(z),$$

where $\varphi \in C^2(\mathbf{R})$, and

$$\hat{a} := \int_E \pi(dx)\lambda(x)\mu(x), \qquad \hat{b}^2 := \int_E \pi(dx)\lambda(x)\sigma^2(x).$$

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Scheme of Proofs

- Step 1: Discrete generator convergence The discrete generator $\mathbb{L}^\varepsilon := \varepsilon^{-1} Q^\varepsilon$ can be written in asymptotic form, for $x \in E_\varepsilon = \varepsilon\mathbb{N}$, $\mathbb{N} := \{0, 1, 2...\}$ and $\varphi \in C_0^2(\mathbb{R})$, as

$$\mathbb{L}^\varepsilon \varphi(x) = \varepsilon^{-1} x(\mu_\varepsilon - 1)\varphi'(x) + \frac{1}{2}[\sigma_\varepsilon^2 x + \varepsilon^{-1} x^2 (\mu_\varepsilon - 1)^2]\varphi''(x) + \theta^\varepsilon(x).$$

Step 2: Tightness

1: compact containment condition

$$\lim_{c\to\infty} \sup_{0<\varepsilon<\varepsilon_0} \mathbb{P}\Big(\sup_{0\le t\le T} |Y_t^\varepsilon| > c \Big) = 0, \qquad (4)$$

2: the inequality

$$\mathbf{E}\,|Y_t^\varepsilon - Y_s^\varepsilon|^2 \le k\,|t-s|$$

(Liptser (1994)).

For Random Environment

The corresponding generator is:

$$\mathbb{L}^{\varepsilon} := \varepsilon^{-1} Q + A(x) + \theta^{\varepsilon}(x)$$

where $A(x)$ is the main part of BP operator and θ^{ε} a negligible operator.
We solve the SPP (see Koroliuk & Limnios (2005)):

$$\mathbb{L}^{\varepsilon} \varphi^{\varepsilon}(u, x) := \mathbb{L}\varphi(u) + \varepsilon \theta_1^{\varepsilon}(x, u).$$

on test functions $\varphi^{\varepsilon}(u, x) = \varphi(u) + \varepsilon \varphi_1(u, x)$.

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Concluding remarks

- The proposed random evolution approach works well;
- Many other cases in RE can be worked in a similar way: REBP with immigration, in semi-Markov RE, More implied results in RE case, etc.
- Normal deviation results can be obtained.
- Results for BPRW are also to be considered.

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Thank you !