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Continuous-state branching processes with dependent immigration

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Based on L ('18+) and L–Zhang ('18+).

1. Branching process with immigration

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\{\xi_{n,i}\}$ be a class of $\mathbb{N}$-valued i.i.d. random variables. Given $X_0 = k \in \mathbb{N}$, we can define a Galton–Watson process $\{X_n\}$ by

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i}, \qquad n \geq 1. \tag{1}$$

Let $\{\eta_n\}$ be a class of $\mathbb{N}$-valued i.i.d. random variables independent of $\{\xi_{n,i}\}$. Given $Y_0 = k \in \mathbb{N}$, we can define a Galton–Watson process with immigration $\{Y_n\}$ by

$$Y_n = \sum_{i=1}^{Y_{n-1}} \xi_{n,i} + \eta_n, \qquad n \geq 1. \tag{2}$$

The above formulations do not work in the continuous-time situation.

- Let $p(j) = \boldsymbol{P}(\xi_{1,1} = j)$. Then $\{X_n\}$ is a Markov chain with one-step transition matrix $Q(i, j) = p * \cdots * p(j)$ (i-fold convolution). The following branching property holds:

$$Q(i_1 + i_2, \cdot) = Q(i_1, \cdot) * Q(i_2, \cdot), \qquad i_1, i_2 \in \mathbb{N}. \tag{3}$$

- Let $\gamma(j) = \boldsymbol{P}(\eta_1 = j)$. Then $\{Y_n\}$ is a Markov chain with one-step transition matrix:

$$P(i, \cdot) := Q(i, \cdot) * \gamma(\cdot), \qquad i \in \mathbb{N}. \tag{4}$$

A continuous-time Markov process with state space $[0,\infty)$ is called a continuous-state branching process if its transition probability $Q_t(x,dy)$ satisfies:

$$Q_t(x_1+x_2,\cdot) = Q_t(x_1,\cdot) * Q_t(x_2,\cdot), \qquad x_1,x_2,t \geq 0. \tag{5}$$

This is the (continuous-time) branching property, which yields the following structures:

$$\int_{[0,\infty)} e^{-\lambda y} Q_t(x,dy) = e^{-xv_t(\lambda)}, \qquad x,t,\lambda \geq 0, \tag{6}$$

where $t \mapsto v_t(\lambda)$ is the positive solution to

$$\frac{d}{dt} v_t(\lambda) = -\phi(v_t(\lambda)), \quad v_t(\lambda) = \lambda. \tag{7}$$

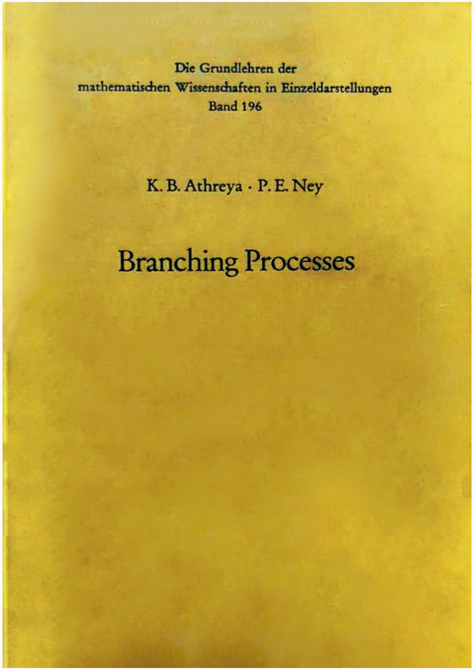
- The function ϕ is the branching mechanism give by

$$\phi(z) = bz + cz^2 + \int_{(0,\infty)} \big(\mathrm{e}^{-zu} - 1 + zu\big) m(\mathrm{d}u), \tag{8}$$

where $c \geq 0$ and b are constants and $(u \wedge u^2)m(\mathrm{d}u)$ is a finite measure on $(0,\infty)$.

References: Feller ('51), Lamperti ('67), Silverstein ('68), etc.

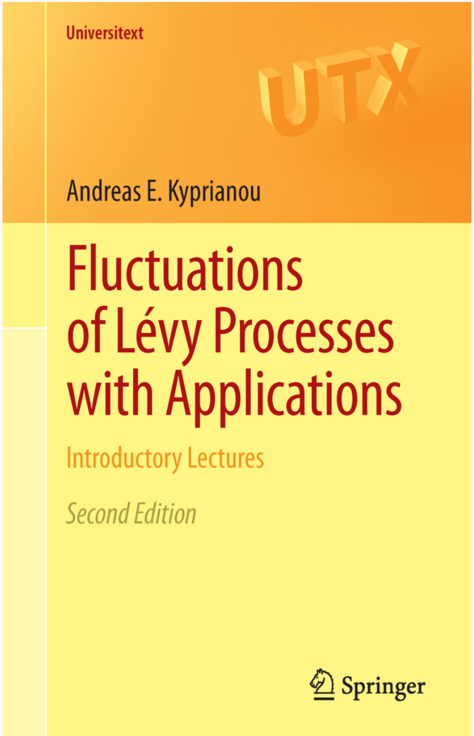
Books discussing CB- and CBI-processes



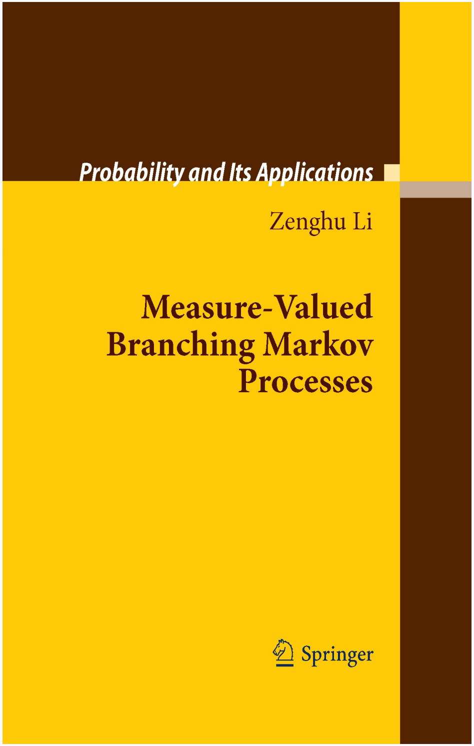
Chapter V. Multi-Type Branching Processes

Chapter VI. Special Processes

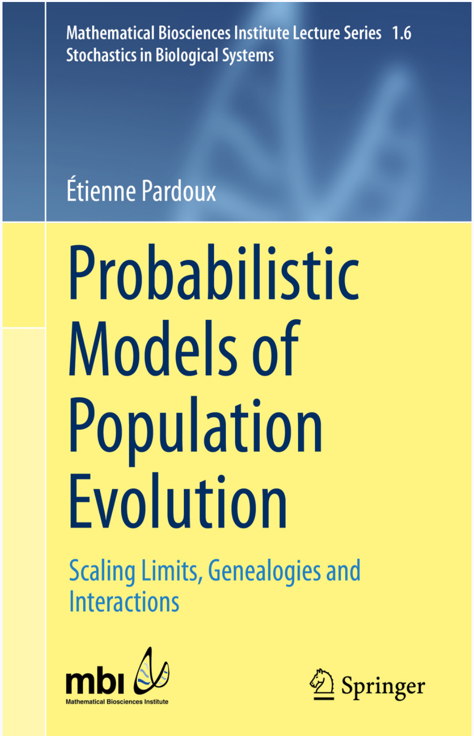
Athreya–Ney (Springer '72) – Section VI.2



Kyprianou (Springer ’06/’14) – Chapter 12



Li (Springer ’11) – Chapter 3 and Section 9.5



Pardoux (Springer ’16) – Chapter 4

A Markov process with state space $[0,\infty)$ is called a continuous-state branching process with immigration if its transition probability $P_t(x,dy)$ has the decomposition:

$$P_t(x,\cdot) = Q_t(x,\cdot) * \gamma_t(\cdot), \qquad x,t \geq 0. \tag{9}$$

where $(\gamma_t)_{t\geq 0}$ is a family of probability distributions on $[0,\infty)$.

Proposition 0 (L '11) *The kernels $(P_t)_{t\geq 0}$ form a semigroup if and only if*

$$\gamma_{r+t} = (\gamma_r Q_t) * \gamma_t, \qquad r,t \geq 0, \tag{10}$$

where

$$(\gamma_r Q_t)(\cdot) = \int_{[0,\infty)} \gamma_r(dx) Q_t(x,\cdot). \tag{11}$$

- The skew-convolution property (10) yields the following structure:

$$\int_{[0,\infty)} e^{-\lambda y}\gamma_t(dy) = \exp\Big\{ -\int_0^t \psi(v_s(\lambda))ds\Big\}. \tag{12}$$

- The function ψ is the immigration mechanism given by

$$\psi(z) = \beta z + \int_{(0,\infty)} (1 - \mathrm{e}^{-zu})\nu(\mathrm{d}u), \tag{13}$$

where $\beta \geq 0$ and $(1\wedge u)\nu(\mathrm{d}u)$ is a finite measure on $(0,\infty)$.

The CBI-process is also known as "CIR-model" in mathematical finance;

e.g, Bernis–Scotti ('18+)., Jiao et al. ('17).

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Alpha-CIR model with branching processes in sovereign interest rate modeling

Ying Jiao[1,2] · Chunhua Ma[3] · Simone Scotti

Abstract We introduce a class of interest rate models, called the α-CIR model, which is a natural extension of the standard CIR model by adding a jump part driven by α-stable Lévy processes with index $\alpha \in (1, 2]$. We deduce an explicit expression for the bond price by using the fact that the model belongs to the family of CBI and affine processes, and analyze the bond price and bond yield behaviors. The α-CIR model allows us to describe in a unified and parsimonious way several recent observations on the sovereign bond market such as the persistency of low interest rates together with the presence of large jumps. Finally, we provide a thorough analysis of the jumps, and in particular the large jumps.

Structures of a biological population:

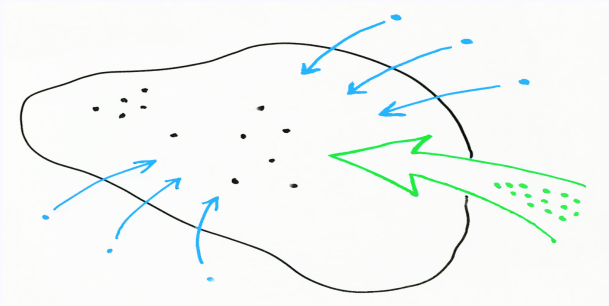
natives + immigrants

(immigrants reproduce in the same way as the natives)

Types of immigration:

individual immigrants + group immigrants

(continuous immigration + discontinuous immigration)



(14)

Problem Construction of immigration models reflecting those structures.

2. Generators of three population models

The generators of the CB- and the CBI-processes are known by the results of Lamperti ('67) and Kawazu–Watanabe ('71); see also Aliev ('85), Aliev–Shchurenkov ('82), Grey ('74), Grimvall ('74).

● A continuous-state branching process (CB-process) without immigration has generator L_0 defined by (determined by the branching mechanism ϕ):

$$L_0 f(x) = x\Big[cf''(x) - bf'(x) + \int_{(0,\infty)} \big(f(x+z) - f(x) - zf'(x)\big)m(\mathrm{d}z)\Big], \quad (15)$$

where the part in [...] is the generator of a spectrally positive Lévy process. (Linear!)

● The continuous-state branching process with immigration (CBI-process) has generator L_1 defined by (determined by the branching mechanism ϕ and the immigration mechanism ψ):

$$L_1 f(x) = L_0 f''(x) + \Big[\beta f'(x) + \int_{(0,\infty)} \big(f(x+z) - f(x)\big)\nu(\mathrm{d}z)\Big], \quad (16)$$

where the part in [...] is the generator of an increasing Lévy process. (Affine!)

● By a continuous-state branching process with dependent immigration (CBDI-process) we mean a continuous-times Markov process with generator L defined by

$$Lf(x) = L_0 f''(x) + h(x)f'(x) + \int_{(0,\infty)} \big(f(x+z) - f(x)\big) q(x,z)\nu(\mathrm{d}z), \quad (17)$$

where (h, q) are positive functions representing the dependent immigration rates.

We assume the following conditions:

- (linear growth condition) there is a constant $K \geq 0$ so that

$$h(x) + \int_0^\infty q(x,z) z\nu(\mathrm{d}z) \leq K(1+x), \qquad x \geq 0;$$

- (Yamada-Watanabe condition) there is an increasing and concave function $u \mapsto r(u)$ on $[0,\infty)$ so that $\int_{0+} r(u)^{-1}\mathrm{d}u = \infty$ and, for $x, y \geq 0$,

$$|h(x) - h(y)| + \int_0^\infty |q(x,z) - q(y,z)| z\nu(\mathrm{d}z) \leq r(|x-y|).$$

This talk: A construction of the CBDI-process.

3. Construction of the CBDI-process

Recall that a CB-process (without immigration) has transition semigroup $(Q_t)_{t\geq 0}$ characterized by

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \qquad t, \lambda, x \geq 0, \tag{18}$$

where $t \mapsto v_t(\lambda)$ is the unique positive solution of

$$\frac{\partial}{\partial t} v_t(\lambda) = -\phi(v_t(\lambda)), \quad v_0(\lambda) = \lambda. \tag{19}$$

The function ϕ is the branching mechanism given by

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda z} - 1 + \lambda z) m(\mathrm{d}z). \tag{20}$$

Observations: (i) $(Q_t)_{t\geq 0}$ is a Feller semigroup; (ii) zero is a trap for the CB-process; (iii) $Q_t(x, \cdot)$ is infinitely divisible distribution.

- The CB-process can be realized in $W := \{$positive càdlàg paths $(w(t))_{t\geq 0}\}$.

- By restricting $(Q_t)_{t\geq 0}$ on $(0, \infty)$, we obtain a sub-Markov semigroup $(Q_t^\circ)_{t\geq 0}$.

Since $Q_t(x,\cdot)$ is infinite divisible, we have the canonical representation:

$$v_t(\lambda) = h_t\lambda + \int_{(0,\infty)} (1-\mathrm{e}^{-\lambda z}) l_t(\mathrm{d}z), \tag{21}$$

where $h_t \geq 0$ and $zl_t(\mathrm{d}z)$ is a finite measure on $(0,\infty)$.

- The family $(l_t)_{t>0}$ is an entrance rule for $(Q_t^\circ)_{t\geq 0}$ in the sense that, for any $t>0$,

$$l_r Q_{t-r}^\circ := \int_{(0,\infty)} l_r(\mathrm{d}x) Q_{t-r}^\circ(x,\cdot) \uparrow l_t \quad \text{as } r\uparrow t. \tag{22}$$

- We construct (directly) the Kuznetsov measure $\boldsymbol{N}_0$ on W so that (Markov property):

$$\begin{aligned} &\boldsymbol{N}_0(w(t_1)\in \mathrm{d}x_1, w(t_2)\in \mathrm{d}x_2, \dots, w(t_n)\in \mathrm{d}x_n) \\ &\quad = l_{t_1}(\mathrm{d}x_1) Q_{t_2-t_1}^\circ(x_1,\mathrm{d}x_2)\cdots Q_{t_n-t_{n-1}}^\circ(x_{n-1},\mathrm{d}x_n). \end{aligned} \tag{23}$$

- In the special case of $h_t \equiv 0$ $(t>0)$, it is the excursion law $\boldsymbol{N}_0 := \lim_{z\downarrow 0} z^{-1}\boldsymbol{Q}_z$, where $\boldsymbol{Q}_z$ is the distribution on W of the CB-process $(x(t))_{t\geq 0}$ with $x(0)=z$.

Recall that $\boldsymbol{Q}_z$ is the distribution on W of the CB-process $(x(t))_{t\geq 0}$ with $x(0) = z$ and $\boldsymbol{N}_0$ is the canonical Kuznetsov measure defined by (23).

Suppose we have the following independent elements:

- $X_t =$ CB-process with generator L_0 given by (15);
- $N_0(\mathrm{d}s, \mathrm{d}u, \mathrm{d}w) =$ Poisson r.m. on $(0, \infty)^2 \times W$ with intensity $\mathrm{d}s\mathrm{d}u\boldsymbol{N}_0(\mathrm{d}w)$;
- $N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w) =$ Poisson r.m. on $(0, \infty)^3 \times W$ with intensity $\mathrm{d}s\nu(\mathrm{d}z)\mathrm{d}u\boldsymbol{Q}_z(\mathrm{d}w)$.

We consider the stochastic integral equation (L '11/'18+, L–Zhang '18+):

$$\begin{aligned} Y_t \;=\; X_t + \int_0^t h(Y_s)h_{t-s}\mathrm{d}s + \int_0^t \int_0^{h(Y_{s-})} \int_W w(t-s) N_0(\mathrm{d}s, \mathrm{d}u, \mathrm{d}w) \\ + \int_0^t \int_0^\infty \int_0^{q(Y_{s-}, z)} \int_W w(t-s) N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w). \end{aligned} \tag{24}$$

Theorem 2 *There is a pathwise unique positive càdlàg solution* $\{Y_t : t \geq 0\}$ *to (29) and the solution is a Markov process with generator* L *defined by (17).*

...... see (14) and (17).

Recall that a CBDI-process has generator L given by ($L_0 =$ generator of CB-process):

$$Lf(x) = L_0 f''(x) + h(x)f'(x) + \int_{(0,\infty)} \big(f(x+z) - f(x)\big)q(x,z)\nu(\mathrm{d}z). \quad (25)$$

Example 1 When $h(x) \equiv \beta$ and $q(x,z) \equiv 1$ are constants, the operator L defined by (25) generates a classical CBI-process.

Example 2 When $h(x) \equiv \beta x$ and $q(x,z) \equiv x$, the operator L defined by (25) generates a CB-process with new branching mechanism

$$\lambda \mapsto \phi(\lambda) - \beta\lambda - \int_0^\infty (1 - \mathrm{e}^{-\lambda z})\nu(\mathrm{d}z).$$

Example 3 Suppose that $x \mapsto G(x)$ be a "good" positive function on $[0,\infty)$. By setting $h(x) = \beta x - G(x)$ and $q(x,z) \equiv 0$ we get (CB-process with competition):

$$Lf(x) = L_0 f(x) + \beta x f'(x) - G(x)f'(x).$$

See Berestycki et al. ('17+), Lambert ('05), Pardoux ('16).

4. Distributions of large jumps

Using the stochastic equations, we can calculate some useful probabilities explicitly.

Theorem 3 (He–L '15) *Let $\{y(t)\}$ be a CB-process with $y(0) = x$. Then, for any $r, t > 0$,*

$$\boldsymbol{P}(s \mapsto y(s) \text{ has no jump larger than } r \text{ by time } t) = \exp\{-x u_r(t)\}, \tag{26}$$

where $t \mapsto u_r(t)$ is the unique solution of

$$\frac{\partial}{\partial t} u_r(t) = m(r, \infty) - \phi_r(u_r(t)), \qquad u_r(0) = 0, \tag{27}$$

where

$$\phi_r(\lambda) = \Big[b + \int_{(r,\infty)} z m(dz)\Big]\lambda + c\lambda^2 + \int_{(0,r]} (e^{-\lambda z} - 1 + \lambda z) m(dz).$$

Corollary 4 *For $r > 0$ satisfying $m(r, \infty) > 0$, we have*

$$\boldsymbol{P}(s \mapsto y(s) \text{ has no jump larger than } r \text{ forever}) = \exp\{-x \phi_r^{-1}(m(r, \infty))\}. \tag{28}$$

Remark We have not seen the counterparts of (26) and (28) in the discrete-time/state setting.

Observations for large jumps

Let $\{x_r(t)\}$ be a CB-process with branching mechanism ϕ_r and $N_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w)$ a Poisson random measure with intensity $1_{\{z>r\}}\mathrm{d}s m(\mathrm{d}z)\mathrm{d}u \boldsymbol{Q}_z(\mathrm{d}w)$ independent of $\{x_r(t)\}$. Then the solution $\{y(t)\}$ of

$$y(t) = x_r(t) + \int_0^t \int_r^\infty \int_0^{y(s-)} \int_W w(t-s) N_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w) \tag{29}$$

is a CB-process with branching mechanism ϕ. Observe that

$$\begin{aligned}
&\Big\{ s \mapsto y(s) \text{ has no jump larger than } r \text{ by time } t \Big\} \\
&= \Big\{ \int_0^t \int_r^\infty \int_0^{y(s-)} \int_W w(t-s) N_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w) = 0 \Big\} \\
&= \Big\{ \int_0^t \int_r^\infty \int_0^{y(s-)} \int_W N_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w) = 0 \Big\} \\
&\qquad \bigcap \Big\{ y(s) = x_r(s) \text{ for } s \in [0, t] \Big\} \\
&= \Big\{ \int_0^t \int_r^\infty \int_0^{x_r(s-)} \int_W N_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w) = 0 \Big\} \\
&\qquad \bigcap \Big\{ y(s) = x_r(s) \text{ for } s \in [0, t] \Big\} \\
&= \Big\{ \int_0^t \int_r^\infty \int_0^{x_r(s-)} \int_W N_r(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}w) = 0 \Big\}.
\end{aligned}$$

5. Proof of existence of the solution

For simplicity, assume $q(x,z) \equiv 0$. Let $Y_0(t) = X_t$ and define inductively

$$Y_k(t) = X_t + \int_0^t h(Y_{k-1}(s)) h_{t-s} \mathrm{d}s + \int_0^t \int_0^{h(Y_{k-1}(s-))} \int_W w(t-s) N_0(\mathrm{d}s, \mathrm{d}u, \mathrm{d}w).$$

Then $\left(\int_a^b = - \int_b^a \right)$

$$\begin{aligned} Y_j(t) - Y_k(t) &= \int_0^t \left[h(Y_{j-1}(s)) - h(Y_{k-1}(s)) \right] h_{t-s} \mathrm{d}s \\ &\quad + \int_0^t \int_{h(Y_{k-1}(s-))}^{h(Y_{j-1}(s-))} \int_W w(t-s) N_0(\mathrm{d}s, \mathrm{d}u, \mathrm{d}w) \end{aligned}$$

Observe that $|Y_j(t) - Y_k(t)| \le Z_{j,k}(t)$, where

$$\begin{aligned} Z_{j,k}(t) &= \int_0^t \left| h(Y_{j-1}(s)) - h(Y_{k-1}(s)) \right| h_{t-s} \mathrm{d}s \\ &\quad + \int_0^t \int_{h(Y_{j-1}(s-)) \wedge h(Y_{k-1}(s-))}^{h(Y_{j-1}(s-)) \vee h(Y_{k-1}(s-))} \int_W w(t-s) N_0(\mathrm{d}s, \mathrm{d}u, \mathrm{d}w) \end{aligned}$$

By a moment formula of stochastic integrals,

$$\boldsymbol{E}[Z_{j,k}(t)] = \boldsymbol{E}\Big[\int_0^t \left| h(Y_{j-1}(s)) - h(Y_{k-1}(s)) \right| \Big(h_{t-s} + \int_W w(t-s) \boldsymbol{N}_0(\mathrm{d}w) \Big) \mathrm{d}s \Big].$$

It follows that

$$\begin{aligned} \boldsymbol{E}[Z_{j,k}(t)] &= \boldsymbol{E}\Big[\int_0^t \Big|h(Y_{j-1}(s)) - h(Y_{k-1}(s))\Big| \mathrm{e}^{-b(t-s)} \mathrm{d}s\Big] \\ &\le \mathrm{e}^{|b|t} \boldsymbol{E}\Big[\int_0^t r(|Y_{j-1}(s) - Y_{k-1}(s)|)\mathrm{d}s\Big] \quad \text{(Yamada–Watanabe cond.)} \\ &\le \mathrm{e}^{|b|t} \boldsymbol{E}\Big[\int_0^t r(Z_{j-1,k-1}(s))\mathrm{d}s\Big] \le \mathrm{e}^{|b|t} \int_0^t r(\boldsymbol{E}[Z_{j-1,k-1}(s)])\mathrm{d}s. \end{aligned}$$

Let $R_n(t) = \sup_{j,k\ge n} \boldsymbol{E}[Z_{j,k}(t)]$ and $R(t) = \lim_{n\to\infty} R_n(t)$. Then

$$R_n(t) \le \mathrm{e}^{|b|t} \int_0^t r(R_{n-1}(s))\mathrm{d}s \Rightarrow R(t) \le \mathrm{e}^{|b|t} \int_0^t r(R(s))\mathrm{d}s \Rightarrow R(t) = 0,$$

and so

$$\lim_{j,k\to\infty} \boldsymbol{E}[|Y_j(t) - Y_k(t)|] \le \lim_{j,k\to\infty} \boldsymbol{E}[Z_{j,k}(t)] = 0.$$

With some additional work, we show

$$\lim_{j,k\to\infty} \boldsymbol{E}\Big[\sup_{0\le s\le t} |Y_j(s) - Y_k(s)|\Big] = 0.$$

Then $\{Y_k(t) : t \ge 0\}$, $k = 1, 2, \cdots$ is a Cauchy sequence. □

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Thank You!