Fluid limits with random initial conditions.

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Summary

- Stochastic non linear dynamics with small noise (X_t^ε)
- AND starting at ε near zero (zero is absorbing state)
- By classical result (Kurtz, Freidlin-Wentzell) approximation on a finite time interval $[0, T]$ is the deterministic function, solves ODE, which starts at zero. Hence it is zero for all times
- Take $T_\varepsilon \to \infty$ $(c \log(1/\varepsilon))$.
  Then $X^\varepsilon_{T_\varepsilon+t}$ converges to the fluid limit but with a new initial condition
- The initial condition is of the form $H(W)$, where
  $H(x)$ is a function obtained from the deterministic equation
  W is a random variable arising in the linear stochastic approximation

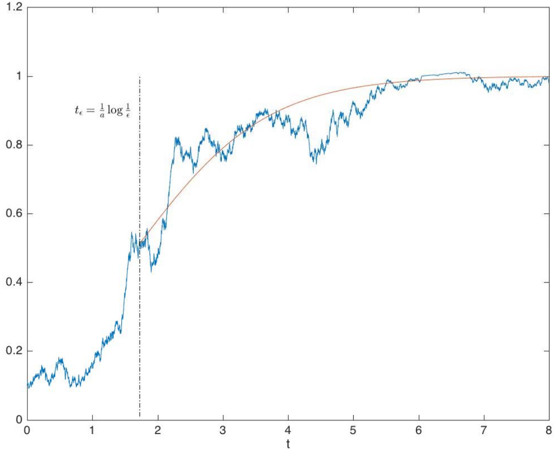

Figure : Fluid approximation

$$dX_t^{\varepsilon} = f(X_t^{\varepsilon})dt + \sqrt{\varepsilon\sigma(X_t^{\varepsilon})}dB_t, \quad t \geq 0 \tag{1}$$

where f, σ are twice continuously differentiable functions with bounded second derivatives.

$f(0) = \sigma(0) = 0$, and $f'(0) > 0$ and $\sigma'(0) > 0$, which makes zero an unstable fixed point of (1) as well as of the ode obtained by removing the stochastic perturbation

$$\frac{dx_t}{dt} = f(x_t), \quad t \geq 0. \tag{2}$$

$\sigma(\cdot)$ is assumed to be bounded and $f(\cdot)$ satisfies the following drift condition:

$$(y-x)\big(f(y) - f(x)\big) \leq f'(0)(y-x)^2, \quad x, y \geq 0. \tag{3}$$

Smoothness of the coefficients and the drift condition (3) are sufficient for existence of the unique strong solution of (1) for any initial point $X_0^\varepsilon > 0$.
Similarly the deterministic equation (2) admits unique continuous solution subject to any $x_0 \geq 0$.
A classical result e.g. Freidlin and Wentzell: the effect of noise on any *fixed* time interval $[0, T]$ is negligible as $\varepsilon \to 0$.

Theorem

Let X_t^ε *satisfy* (1) *and* $X_0^\varepsilon \xrightarrow[\varepsilon\to 0]{P} x_0 \geq 0$, *then for any* T

$$\sup_{t\leq T} |X_t^\varepsilon - x_t| \xrightarrow[\varepsilon\to 0]{P} 0,$$

where x_t *is the solution of* (2) *subject to the initial condition* x_0.

Since zero is a fixed point of the deterministic dynamics (2), this theorem implies that the solution of (1), started from a small positive initial condition $X_0^\varepsilon = \varepsilon > 0$, converges to zero on any fixed bounded interval

$$\sup_{t \le T} \left| X_t^\varepsilon \right| \xrightarrow[\varepsilon \to 0]{P} 0, \qquad \forall T \ge 0.$$

On the other hand, since the fixed point is unstable and the initial condition is nonzero, with positive probability, the trajectory X_t^ε is pushed out of the vicinity of the origin and, after sufficiently large period of time, may reach a significant magnitude.

Theorem

Let $X_0^\varepsilon = \varepsilon > 0$ *and define* $T_\varepsilon := \frac{1}{f'(0)} \log \frac{1}{\varepsilon}$*. Then for any* $T > 0$,

$$\sup_{t \in [0,T]} \left| X^\varepsilon_{T_\varepsilon + t} - x_t \right| \xrightarrow[\varepsilon \to 0]{P} 0, \tag{4}$$

where x_t *is the solution of* (2) *subject* $x_0 = H(W)$.

$$H(x) = \lim_{t \to \infty} \phi_t\left(x e^{-f'(0)t}\right), \quad x \geq 0, \tag{5}$$

where $\phi_t(x)$ *is the flow of* (2).

$$W := \lim_{t \to \infty} e^{-f'(0)t} Y_t,$$

$$Y_t = 1 + \int_0^t f'(0) Y_s ds + \int_0^t \sqrt{\sigma'(0) Y_s} dB_s. \tag{6}$$

- W has the compound Poisson distribution with rate $2a$ and exponentially distributed jumps with mean $1/(2a)$.
- $P(x_0 = 0) = P(W = 0) = e^{-2a} > 0$.
  This corresponds to the event on which the process X_t^ε is absorbed at zero in a finite time. On the event $\{W > 0\}$, the trajectories converge to a nontrivial curve, whose initial point is random.
- This type of randomness was observed in biological models of sweeps (G. Martin and A. Lambert. *Theor. Popul. Biol.* 2015. A simple, semi-deterministic approximation to the distribution of selective sweeps in large populations).

Wright-Fisher model with selection

$$dX_t^\varepsilon = aX_t^\varepsilon(1 - X_t^\varepsilon)dt + \sqrt{\varepsilon}\sqrt{X_t^\varepsilon(1 - X_t^\varepsilon)}dB_t, \qquad (7)$$

$f(x) = ax(1-x)$ and $\sigma(x) = x(1-x)$.
The flow of $x_t' = ax_t(1-x_t)$ is

$$\phi_t(x) = \frac{xe^{at}}{1 - x + xe^{at}},$$

it follows that

$$H(x) = \lim_{t\to\infty} \phi_t(xe^{-at}) = \frac{x}{1+x}.$$

Hence the random initial condition is given by

$$x_0 = \frac{W}{W+1}.$$

Balancing selection model

$$dX_t^\varepsilon = aX_t^\varepsilon(1 - X_t^\varepsilon)(1 - 2X_t^\varepsilon)dt + \sqrt{\varepsilon}\sqrt{X_t^\varepsilon(1 - X_t^\varepsilon)}dB_t,$$

The fluid limit is given by $x_t' = ax_t(1 - x_t)(1 - 2x_t), \quad t \geq 0,$ which generates the flow

$$\phi_t(x) = \frac{1}{2} - \frac{1}{2}\frac{1 - 2x}{\sqrt{4x(1 - x)(e^{at} - 1) + 1}}, \quad x \in (0, \tfrac{1}{2}).$$

$$H(x) = \lim_{t\to\infty} \phi_t(xe^{-at}) = \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{4x + 1}}.$$

Hence the random initial condition

$$x_0 = \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{4W + 1}}.$$

Without loss of generality we fix the normalization $\sigma'(0) = 1$ and denote $a := f'(0) > 0$. The main step in the proof is to establish convergence (4) at $t = 0$

$$X^{\varepsilon}_{T_\varepsilon} \xrightarrow[\varepsilon \to 0]{P} H(W). \tag{8}$$

The rest of the proof follows by a change of time. Indeed, by letting $\widetilde{X}^{\varepsilon}_t = X^{\varepsilon}_{T_\varepsilon + t}$,
and $\widetilde{B}_t = B_{T_\varepsilon + t} - B_{T_\varepsilon}$ we obtain from (1)

$$\widetilde{X}^{\varepsilon}_t = \widetilde{X}^{\varepsilon}_0 + \int_0^t f(\widetilde{X}^{\varepsilon}_s) ds + \int_0^t \sqrt{\varepsilon \sigma(\widetilde{X}^{\varepsilon}_s)} d\widetilde{B}_s,$$

and the result follows from (8) by Theorem 1.

The proof consists of a number of steps.

- A nontrivial limit $H(x) = \lim_{t\to\infty} \phi_t(xe^{-at})$ exists if $\frac{1}{f(x)} - \frac{1}{ax}$ is integrable at zero, with convergence uniform on compacts.
- We take Feller branching diffusion

$$Y_t = 1 + \int_0^t aY_s ds + \int_0^t \sqrt{Y}_s dB_s, \quad t \geq 0, \tag{9}$$

driven by the same Brownian motion as in (1).

$$W_t := e^{-at} Y_t \to_{t\to\infty} W := 1 + \int_0^\infty e^{-as} \sqrt{Y_s} dB_s.$$

Laplace transform of Y_t is known, hence of W.

Let $t_c = \frac{c}{a}\log\frac{1}{\varepsilon}$ with $c \in (1/2, 1)$ and $t_1 = T_\varepsilon = \frac{1}{a}\log\frac{1}{\varepsilon}$.

- Let $\hat{X}^\varepsilon_t := \varepsilon^{-1} X^\varepsilon_t$. Then (Yamada-Watanabe type approximation)

$$\hat{X}^\varepsilon_t \xrightarrow[\varepsilon\to 0]{L^1} Y_t, \quad \forall\, t \geq 0.$$

- $W^\varepsilon_{t_c} = e^{-at_c}\hat{X}^\varepsilon_{t_c} \to W$ as $\varepsilon \to 0$ in L^1.

- $X^{\varepsilon}_{t_c} = W^{\varepsilon}_{t_c} e^{-a(t_1 - t_c)}$. Hence

$$\phi_{t_c,t_1}(X^{\varepsilon}_{t_c}) = \phi_{t_c,t_1}\left(W^{\varepsilon}_{t_c} e^{-a(t_1-t_c)}\right)$$

$$= \phi_{t_1-t_c}\left(W^{\varepsilon}_{t_c} e^{-a(t_1-t_c)}\right) \to H(W).$$

- $\Phi_{s,t}(x)$ flow of the sde (1)

$$\Phi_{t_c,t_1}(X^{\varepsilon}_{t_c}) - \phi_{t_c,t_1}(X^{\varepsilon}_{t_c}) \xrightarrow[\varepsilon \to 0]{L^2} 0.$$

- It now follows

$$X^{\varepsilon}_{T_\varepsilon} = X^{\varepsilon}_{t_1} = \Phi_{t_c,t_1}(X^{\varepsilon}_{t_c}) \to H(W).$$

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Birth-Death processes

Birth-Death process $Z^K = (Z_t^K; t \geq 0)$ on $\mathbb{Z}_+$ with per capita rates:

$$\lambda_K(z) := \lambda - (\lambda - \mu) g_1(z/K) \qquad \text{(birth)}$$
$$\mu_K(z) := \mu + (\lambda - \mu) g_2(z/K) \qquad \text{(death)}$$

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- The "linear" part: $\lambda > \mu \geq 0$ are real constants
- The "nonlinear" part: $g_1(0) = g_2(0) = 0$, continuous, increasing, $g = g_1 + g_2$
- Carrying capacity: $K \gg 1$ is a large parameter

A nonstandard fluid limit: the ingredients

- K-dependent time shift

$$t(K) := \frac{1}{\lambda - \mu} \log K$$

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- the martingale limit of the linear branching Y_t with rates λ and μ

$$W = \lim_{t \to \infty} e^{-(\lambda - \mu)t} Y_t,$$

which has exponential distribution with an atom at zero:

$$P(W = 0) = \frac{\mu}{\lambda}$$
$$P(W > t) = \left(1 - \frac{\mu}{\lambda}\right) e^{-(1 - \frac{\mu}{\lambda})t}, \quad t \in [0, \infty)$$

Theorem (Barbour, Chigansky, K. 2016)

For a fixed integer $Z_0 = z \in \mathbb{N}$, the density process $\overline{Z}_t^K = Z_t^K / K$ satisfies

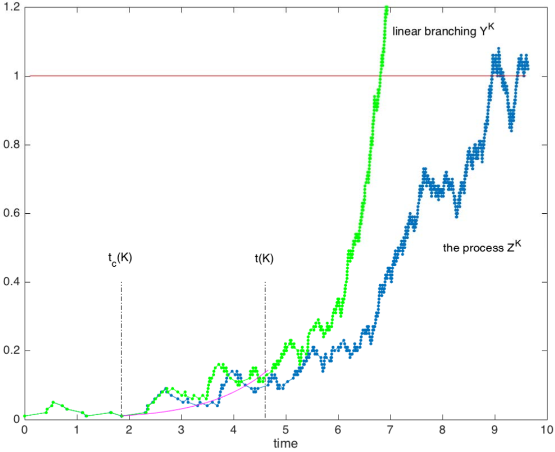
$$\sup_{t\in[0,T]} \left| \overline{Z}^K_{t(K)+t} - x_t \right| \xrightarrow[K\to\infty]{P} 0$$

where x_t is the solution of the ODE

$$\dot{x}_t = (\lambda - \mu) x_t \big(1 - g(x_t)\big)\, dt, \quad t \in [0, T]$$

subject to the random initial condition

$$x_0 := H\Big(\underbrace{W_1 + ... + W_z}_{i.i.d.\ copies\ of\ W}\Big).$$



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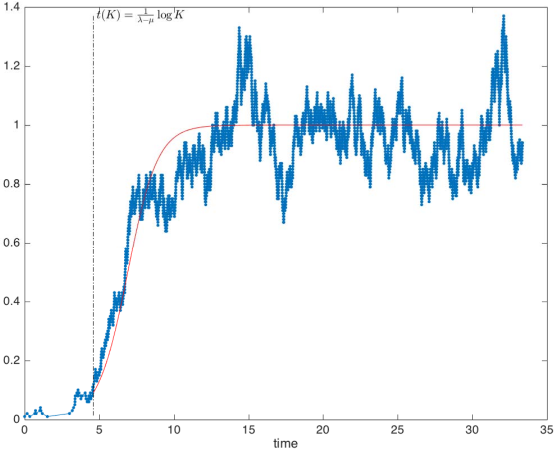

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- V^K is the "slower" linear branching with "frozen" per capita rates

$$\lambda - (\lambda - \mu) g_1(K^{\eta+c}/K) \quad \text{and} \quad \mu + (\lambda - \mu) g_2(K^{\eta+c}/K)$$

$V_t^K \leq Y_t^K \leq U_t^K, \forall t$ $V_t^K \leq Z_t^K \leq U_t^K$ till Z^K hits the level $K^{\eta+c}$

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Thank You