The Galton-Watson process in varying environment –
a stepchild in branching?

Götz Kersting

Goethe Universität, Frankfurt am Main

WBPA 2018, Badajoz,
April 10-13, 2018

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Branching process in varying environment (BPVE).

Generalizes the Galton-Watson process (GWP),
in that the offspring distribution may change in a deterministic fashion from one generation to the next: $f_1, f_2, \ldots$

Studied by Agresti, Jagers, Lindvall, Biggins, De Souza, Lyons, the Badajoz group, Bansaye, . . .

Attained much less attention - in particular as a tool - than the GWP, despite of its natural appearance

Branching process in varying environment (BPVE).

Generalizes the Galton-Watson process (GWP), in that the offspring distribution may change in a deterministic fashion from one generation to the next: $f_1, f_2, \ldots$

Studied by Agresti, Jagers, Lindvall, Biggins, De Souza, Lyons, the Badajoz group, Bansaye, . . .

Attained much less attention - in particular as a tool - than the GWP, despite of its natural appearance

Branching process in varying environment (BPVE).

Generalizes the Galton-Watson process (GWP), in that the offspring distribution may change in a deterministic fashion from one generation to the next: $f_1, f_2, \ldots$

Studied by Agresti, Jagers, Lindvall, Biggins, De Souza, Lyons, the Badajoz group, Bansaye, . . .

Attained much less attention - in particular as a tool - than the GWP, despite of its natural appearance

Related processes:

Special case: Galton-Watson process: $f_1 = f_2 = \cdots = f$.

Defective BPVE,
see Braunsteins & Hautphenne, K. & Minuesa

Branching processes in random environment: $f_1, f_2, \ldots$ make a stationary (i.i.d.) sequence in the space of probability measures.

Related processes:

Special case: Galton-Watson process: $f_1 = f_2 = \cdots = f$.

Defective BPVE,
see Braunsteins & Hautphenne, K. & Minuesa

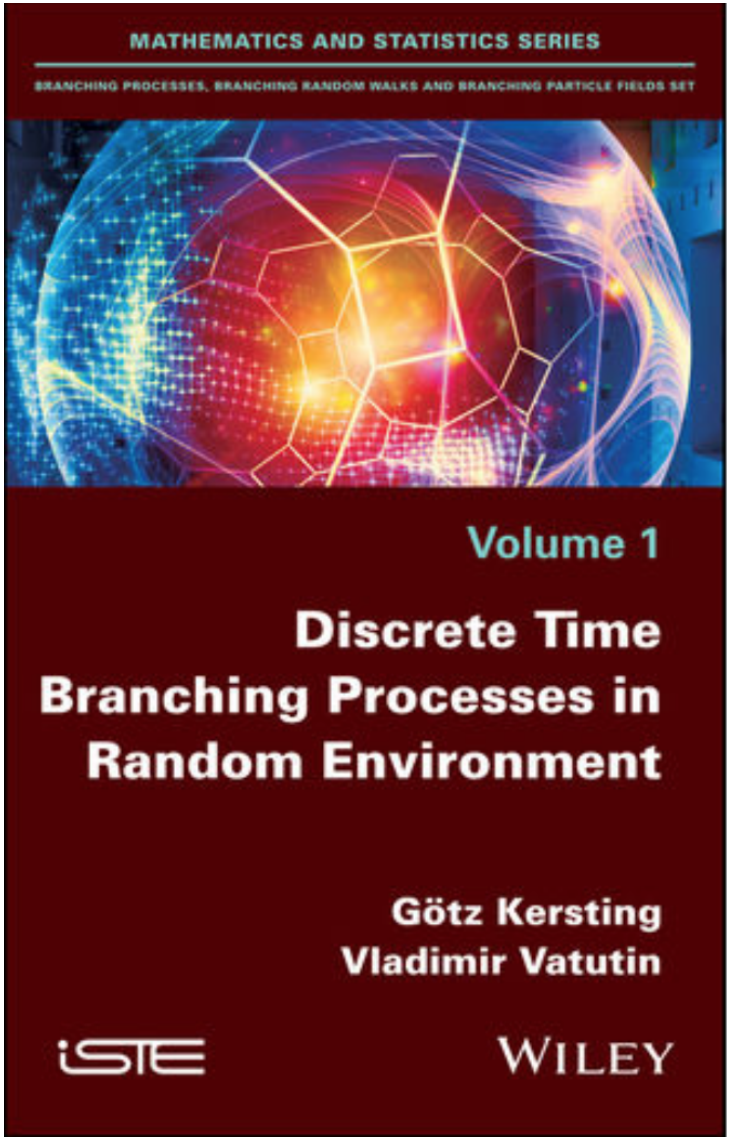
Branching processes in random environment: $f_1, f_2, \ldots$ make a stationary (i.i.d.) sequence in the space of probability measures.

Related processes:

Special case: Galton-Watson process: $f_1 = f_2 = \cdots = f$.

Defective BPVE,
see Braunsteins & Hautphenne, K. & Minuesa

Branching processes in random environment: $f_1, f_2, \ldots$ make a stationary (i.i.d.) sequence in the space of probability measures.


MATHEMATICS AND STATISTICS SERIES
BRANCHING PROCESSES, BRANCHING RANDOM WALKS AND BRANCHING PARTICLE FIELDS SET
Volume 1
Discrete Time Branching Processes in Random Environment
Götz Kersting
Vladimir Vatutin
ISTE
WILEY

THE OBSTACLE

Branching process in varying environment.

A sequence $f_1, f_2, \ldots$ of probability measures on $\mathbb{N}_0$ with weights $f_n[y]$, $n \geq 1$, $y \geq 0$, is called a *varying environment*.

Branching process in varying environment.

The process $Z = (Z_n)_{n\geq 0}$ is called a *BPVE* in the environment $f_1, f_2, \ldots$ if it allows the representation

$$Z_n = \sum_{i=1}^{Z_{n-1}} Y_{in}$$

with independent $\mathbb{N}_0$-valued r.v. Y_{in}, $i, n \geq 1$, also independent of Z_0, such that the Y_{in} are copies of r.v. Y_n with distributions f_n, $i, n \geq 1$.

That is,

$$\mathbf{P}(Y_{in} = y) = f_n[y] \ , \ y \in \mathbb{N}_0 \ .$$

The result of MacPhee and Schuh (1983).

Let, as usual,

$$W_n := \frac{Z_n}{\mathbf{E}[Z_n]}$$

Then the process $(W_n)_{n \geq 0}$ is a non-negative martingale and consequently there is a r.v. W_∞ such that as $n \to \infty$

$$W_n \to W_\infty \text{ a.s.}$$

The result of MacPhee and Schuh (1983).

Theorem. *There are BPVEs such that*

$$\mathbf{P}(Z_\infty = 0) < \mathbf{P}(W_\infty = 0) \ .$$

More precisely, for given $m > 4$ there is a BPVE $(Z_n)_{n\geq 0}$ such that

$$\mathbf{E}[Z_n] \sim am^n$$

for some $0 < a < \infty$, whereas both events

$$0 < \lim_{n\to\infty} \frac{Z_n}{2^n} < \infty \qquad \textit{and} \qquad 0 < \lim_{n\to\infty} \frac{Z_n}{m^n} < \infty$$

have strictly positive probability.

The result of MacPhee and Schuh (1983).

Theorem. *There are BPVEs such that*

$$\mathbf{P}(Z_\infty = 0) < \mathbf{P}(W_\infty = 0) \ .$$

More precisely, for given $m > 4$ *there is a BPVE* $(Z_n)_{n \geq 0}$ *such that*

$$\mathbf{E}[Z_n] \sim a m^n$$

for some $0 < a < \infty$, *whereas both events*

$$0 < \lim_{n \to \infty} \frac{Z_n}{2^n} < \infty \qquad \textit{and} \qquad 0 < \lim_{n \to \infty} \frac{Z_n}{m^n} < \infty$$

have strictly positive probability.

Idea of proof:

Set

$$\mathbf{P}(Y_n = k) = \begin{cases} 1 - 4^{-n} & \text{for } k = 2 \\ 4^{-n} & \text{for } k = 2 + (m-2)4^n \end{cases}$$

Borel-Cantelli gives that the event

$$\{Z_n = 2^n \text{ for all } n\}$$

has positive probability.

Idea of proof:

Set

$$\mathbf{P}(Y_n = k) = \begin{cases} 1 - 4^{-n} & \text{for } k = 2 \\ 4^{-n} & \text{for } k = 2 + (m-2)4^n \end{cases}$$

Borel-Cantelli gives that the event

$$\{Z_n = 2^n \text{ for all } n\}$$

has positive probability.

Work of Agresti, Jagers, Biggins, De Souza, Lyons ... didn't lead to a clear CLASSIFICATION of BPVE, as we appreciate it in case of the GWP.

Russell Lyons remarks in his paper "Random walks, capacities and percolation on trees" (AP, 1992)

"The pathologies ... are possible when the condition

$$\sup_n \|Y_n\|_\infty < \infty \text{ a.s.}$$

(that is uniformly bounded offspring numbers a.s.) is relaxed even a slightest bit."

Work of Agresti, Jagers, Biggins, De Souza, Lyons ... didn't lead to a clear CLASSIFICATION of BPVE, as we appreciate it in case of the GWP.

Russell Lyons remarks in his paper "Random walks, capacities and percolation on trees" (AP, 1992)

"The pathologies ... are possible when the condition

$$\sup_n \|Y_n\|_\infty < \infty \text{ a.s.}$$

(that is uniformly bounded offspring numbers a.s.) is relaxed even a slightest bit."

Are BPRE useless for applications because of their unclear appearence?

Or is there a condition, which

- applies for an overwhelming, generic portion of the BPVEs,
- eliminates pathological behaviour,
- allows for a classification of BPVEs along the lines of GWPs?

Yes, there is!

Are BPRE useless for applications because of their unclear appearence?

Or is there a condition, which

- applies for the overwhelming, generic portion of BPVEs,
- eliminates pathological behaviour,
- allows for a classification of BPVEs along the lines of GWPs?

Yes, there is!

Are BPRE useless for applications because of their unclear appearence?

Or is there a condition, which

- applies for an overwhelming, generic portion of the BPVEs,
- eliminates pathological behaviour,
- allows for a classification of BPVEs along the lines of GWPs?

Yes, there is!

THE REGULARITY ASSUMPTIONS

The regularity assumption (A)

$$\exists\, c < \infty \,\forall\, n \geq 1 :$$

$$\mathbf{E}[Y_n^2; Y_n \geq 2] \leq c\,\mathbf{E}[Y_n; Y_n \geq 2] \cdot \mathbf{E}[Y_n \mid Y_n \geq 1] < \infty \;.$$

A stronger uniformity assumption (B)

$$\forall \varepsilon > 0 \;\exists c_\varepsilon < \infty \;\forall n \geq 1 :$$

$$\mathbf{E}\Big[Y_n^2; Y_n > c_\varepsilon(1 + \mathbf{E}[Y_n])\Big] \leq \varepsilon \mathbf{E}[Y_n^2; Y_n \geq 2]$$

The regularity assumption (A)

$$\exists\, c < \infty\, \forall\, n \geq 1 :$$

$$\mathbf{E}[Y_n^2; Y_n \geq 2] \leq c\,\mathbf{E}[Y_n; Y_n \geq 2] \cdot \mathbf{E}[Y_n \mid Y_n \geq 1] < \infty \; .$$

A stronger uniformity assumption (B)

$$\forall \varepsilon > 0\; \exists c_\varepsilon < \infty\; \forall n \geq 1 :$$

$$\mathbf{E}\Big[Y_n^2; Y_n > c_\varepsilon(1 + \mathbf{E}[Y_n])\Big] \leq \varepsilon \mathbf{E}[Y_n^2; Y_n \geq 2]$$

A still stronger, handy L^3-assumption

$$\exists c' < \infty \ \forall n \geq 1 :$$

$$\mathbf{E}[Y_n(Y_n - 1)(Y_n - 2)] \leq c' \mathbf{E}[Y_n(Y_n - 1)] \ (1 + \mathbf{E}[Y_n])$$

Examples:

(i) $Y_n \leq c'$ a.s. for all $n \geq 1$.

(ii) Poisson-variables Y_n with arbitrary parameters λ_n.

(iii) . . .

A still stronger, handy L^3-assumption

$$\exists c' < \infty \ \forall n \geq 1 :$$

$$\mathbf{E}[Y_n(Y_n - 1)(Y_n - 2)] \leq c' \mathbf{E}[Y_n(Y_n - 1)] \ (1 + \mathbf{E}[Y_n])$$

Examples:

(i) $Y_n \leq c'$ a.s. for all $n \geq 1$.

(ii) Poisson-variables Y_n with arbitrary parameters λ_n.

(iii) . . .

THE RESULTS

We proceed along the lines of Galton-Watson processes.

Let for Z_n

$$\mu_n := \mathbf{E}[Z_n]$$

and for Y_n

$$\nu_n := \frac{\mathbf{E}[Y_n(Y_n - 1)]}{\mathbf{E}[Y_n]^2} \ , \ \rho_n := \frac{\mathbf{Var}[Y_n]}{\mathbf{E}[Y_n]^2} \ .$$

Also write

$$q := \mathbf{P}(Z_\infty = 0)$$

for the probability of extinction.

Theorem 1A: a.s. extinction. *Assume (A). Then the following conditions are equivalent:*

(i) $q = 1$,

(ii) $\mathbf{E}[Z_n] = o\left(\sqrt{\mathbf{Var}[Z_n]}\right)$ *as* $n \to \infty$,

(iii) $\sum_{k=1}^{\infty} \frac{\rho_k}{\mu_{k-1}} = \infty$,

(iv) $\mu_n \to 0$ *and/or* $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} = \infty$

Agresti (1975), R. Lyons (1992)

Theorem 1B: survival. *Assume (A). Then the following conditions are equivalent:*

(v) $q < 1$,

(vi) $\sqrt{\mathbf{Var}[Z_n]} = O(\mathbf{E}[Z_n])$ *as* $n \to \infty$,

(vii) $\sum_{k=1}^{\infty} \frac{\rho_k}{\mu_{k-1}} < \infty$,

(viii) $\exists\, 0 < r \leq \infty : \mu_n \to r$ *and* $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} < \infty$

Recall

$$W_\infty := \lim_{n\to\infty} \frac{Z_n}{\mathbf{E}[Z_n]} \text{ a.s.}$$

Theorem 2: supercritical case. *Assume (A). Then we have:*

(i) *If* $\mathbf{P}(Z_\infty = 0) = 1$, *then* $W_\infty = 0$ *a.s.*

(ii) *If* $\mathbf{P}(Z_\infty = 0) < 1$, *then*

$$\mathbf{E}[W_\infty] = 1$$

and

$$\mathbf{P}(W_\infty = 0) = \mathbf{P}(Z_\infty = 0)\ .$$

D'Souza, Biggins (1992), Goettge (1976)

Theorem 3: subcritical case. *Let (A) be satisfied and let* $q = 1$*. Then these conditions are equivalent:*

(i) *for all* $\varepsilon > 0$ *there is a* $c < \infty$ *such that* $\mathbf{P}(Z_n > c \mid Z_n > 0) \leq \varepsilon$ *for all* $n \geq 0$,

(ii) *there is a* $c > 0$ *such that* $c\mu_n \leq \mathbf{P}(Z_n > 0) \leq \mu_n$ *for all* $n \geq 0$, *or, what amounts to the same thing,*

$$\sup_{n \geq 0} \mathbf{E}[Z_n \mid Z_n > 0] < \infty \ ,$$

(iii) $\displaystyle\sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}} = O\Big(\frac{1}{\mu_n}\Big)$ *as* $n \to \infty$

Theorem 4: critical case. *Let (B) be satisfied and let* $q = 1$*. Assume that*

$$\frac{1}{\mu_n} = o\Big(\sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}}\Big)$$

as $n \to \infty$*. Then ("Kolmogorov's asymptotic")*

$$\mathbf{P}(Z_n > 0) \sim 2\Big(\sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}}\Big)^{-1}$$

as $n \to \infty$*.*

Moreover ("Yaglom limit"), setting

$$a_n := \frac{\mu_n}{2} \sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}} \ , \ n \geq 1 \ ,$$

then $a_n \to \infty$ *and the distribution of* Z_n/a_n *conditioned on the event* $Z_n > 0$ *converges to a standard exponential distribution.*

Theorem 4: critical case. *Let (B) be satisfied and let* $q = 1$*. Assume that*

$$\frac{1}{\mu_n} = o\Big(\sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}}\Big)$$

as $n \to \infty$*. Then ("Kolmogorov's asymptotic")*

$$\mathbf{P}(Z_n > 0) \sim 2\Big(\sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}}\Big)^{-1}$$

as $n \to \infty$*.*

Moreover ("Yaglom limit"), setting

$$a_n := \frac{\mu_n}{2} \sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}} \ , \ n \geq 1 \ ,$$

then $a_n \to \infty$ *and the distribution of* Z_n/a_n *conditioned on the event* $Z_n > 0$ *converges to a standard exponential distribution.*

Jagers (1974), Bhattacharya, Perlman (2017)

THE CLASSIFICATION

Example: $0 < \inf_k \nu_k \le \sup_k \nu_k < \infty$ for all $k \geq 1$.

supercritical, if $$\sum_{k=0}^{\infty} \frac{1}{\mu_k} < \infty$$ ("at least linear growth") ,

critical, if $$\sum_{k=0}^{\infty} \frac{1}{\mu_k} = \infty \quad \text{and} \quad \frac{1}{\mu_n} = o\Big(\sum_{k=0}^{n-1} \frac{1}{\mu_k}\Big) \quad ,$$

subcritical, if $$\sum_{k=0}^{n-1} \frac{1}{\mu_k} = O\Big(\frac{1}{\mu_n}\Big)$$ ("at least exp. decay") *supercritical*, if $\lim_{n\to\infty} \mu_n = \infty$ and $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} < \infty$,

asy. degenerate, if $0 < \lim_{n\to\infty} \mu_n < \infty$ and $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} < \infty$,

critical, if $\sum_{k=1}^{\infty} \frac{\nu_k}{\mu_{k-1}} = \infty$ and $\frac{1}{\mu_n} = o\Big(\sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}}\Big)$,

subcritical, if $\lim_{n\to\infty} \mu_n = 0$ and $\sum_{k=1}^{n} \frac{\nu_k}{\mu_{k-1}} = O\Big(\frac{1}{\mu_n}\Big)$.

THE APPROACH

Let for a probability measure f on $\mathbb{N}_0$ with weights $f[z]$

$$f(s) := \sum_{z=0}^{\infty} s^z f[z] \ , \ 0 \leq s \leq 1$$

and $\varphi(s)$ given by

$$\frac{1}{1 - f(s)} = \frac{1}{f'(1)(1-s)} + \varphi(s) \ , \ 0 \leq s < 1$$

Then, due to convexity,

$$\varphi(s) \geq 0 \ .$$

Z_n has the generating function

$$f_{0,n} := f_1 \circ \cdots \circ f_n \ .$$

It follows

$$\frac{1}{1 - f_{0,n}(s)} = \frac{1}{f_1'(1)(1 - f_{1,n}(s))} + \varphi_1(f_{1,n}(s))$$

and via iteration

$$\frac{1}{1 - f_{0,n}(s)} = \frac{1}{\mu_n(1 - s)} + \sum_{k=1}^{n} \frac{\varphi_k(f_{k,n}(s))}{\mu_{k-1}} \ .$$

Example: Critical Galton-Watson process

$$\frac{1}{1-f_{0,n}(s)} = \frac{1}{1-s} + \sum_{k=1}^{n} \varphi(f_{k,n}(s))$$

$$\frac{1}{\mathbf{P}(Z_n > 0)} = 1 + \sum_{k=1}^{n} \varphi\Big(\mathbf{P}(Z_{k+1} = 0)\Big)$$

$$\varphi(1-) = \frac{\nu}{2}$$

$$\mathbf{P}(Z_n > 0) \sim \frac{2}{\nu n}$$

Lemma. *Assume* $f''(1) < \infty$ *for a generating function* f*. Then for* $0 \leq s < 1$

$$\varphi(s) \leq 2\varphi(1-) \ .$$

Also

$$\varphi(1-) = \frac{f''(1)}{2f'(1)^2} \ .$$

(Geiger, K. (2001), Agresti (1975),)

Lemma. *Assume* $f''(1) < \infty$ *for a generating function* f. *Then for* $0 \le s < 1$

$$\frac{1}{2}\varphi(0) \le \varphi(s) \le 2\varphi(1-) \ .$$

Also

$$\varphi(1-) = \frac{f''(1)}{2f'(1)^2} \ .$$

(Geiger, K. (2001), Agresti (1975), K. (2016))

Assumption (A) is a reformulation of the assumption

$$\varphi_n(0) \ge c_1 \varphi_n(1-) \ .$$

Lemma. *Assume* $f''(1) < \infty$ *for a generating function* f. *Then for* $0 \leq s < 1$

$$\frac{1}{2}\varphi(0) \leq \varphi(s) \leq 2\varphi(1-) \ .$$

Also

$$\varphi(1-) = \frac{f''(1)}{2f'(1)^2} \ .$$

(Geiger, K. (2001), Agresti (1975), K. (2016))

Assumption (A) is a reformulation of the assumption

$$\varphi_n(0) \geq c_1 \varphi_n(1-) \ .$$

Sketch of proof:

Let $m = \varphi'(1)$.

$$\varphi(s) = \frac{m(1-s) - (1-f(s))}{m(1-s)(1-f(s))}$$

$$= \frac{\sum_{y=1}^{\infty} f[y]\Big((y-1) + (y-2)s + \cdots + s^{y-2}\Big)}{m \cdot \sum_{z=1}^{\infty} f[z](1 + s + \cdots + s^{z-1})} \, .$$

φ is neither increasing nor decreasing in general.

Lemma. *Let* g_1, g_2 *be probability measures on* $\mathbb{N}_0$ *such that*

$$\frac{g_2[y]}{g_1[y]} \leq \frac{g_2[z]}{g_1[z]} \text{ for all } y < z \ .$$

Also let $\alpha : \mathbb{N}_0 \to \mathbb{R}$ *be a non-decreasing function. Then*

$$\sum_{y=0}^{\infty} \alpha(y) g_1[y] \leq \sum_{y=0}^{\infty} \alpha(y) g_2[y] \ .$$

Consider for $0 < s \leq 1$ and $r \in \mathbb{N}_0$ the probability measures

$$g_s[y] = \frac{s^{r-y}}{1+s+\cdots+s^r}\ , \quad 0 \leq y \leq r\ .$$

We obtain that

$$\sum_{y=0}^{r} y g_s[y] = \frac{s^{r-1}+2s^{r-2}+\cdots+r}{1+s+\cdots+s^r}$$

is a decreasing function in s.

It follows for $0 \leq s \leq 1$

$$\frac{r}{2} \leq \frac{r+(r-1)s+\cdots+s^{r-1}}{1+s+\cdots+s^r} \leq r\ .$$

Thus let

$$\psi(s) := \frac{\sum_{y=1}^{\infty} f[y](y-1)(1+s+\cdots+s^{y-1})}{\sum_{z=1}^{\infty} f[z](1+s+\cdots+s^{z-1})} \ .$$

Check that we may apply the lemma to g_s, $0 \leq s \leq 1$, given by its weights

$$g_s[y] := \frac{f[y](1+s+\cdots+s^{y-1})}{\sum_{z=1}^{\infty} f[z](1+s+\cdots+s^{z-1})} \ , \quad y \geq 1 \ .$$

Thus ψ is increasing. □