Extinction in block lower Hessenberg branching processes with countably many types

or

The probabilities of extinction in a branching random walk on a strip

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Joint work with Peter Braunsteins

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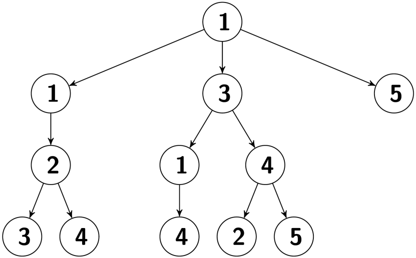
Multi-type Galton-Watson process

- Each individual has a type i in a countable type set $\mathcal{X} \equiv \mathbb{N}$
- The process initially contains a single individual of type φ_0
- Each individual lives for a single generation
- At death, individuals of type i have children according to the progeny distribution : $p_i(\boldsymbol{r}) : \boldsymbol{r} = (r_1, r_2, \ldots)$, where

  $p_i(\boldsymbol{r})$ = probability that a type i gives birth to r_1 children of type 1, r_2 children of type 2, etc.
- All individuals are independent

Multi-type Galton-Watson process



Population size : $\boldsymbol{Z}_n = (Z_{n1}, Z_{n2}, \ldots)$, $n \in \mathbb{N}_0$, where

Z_{ni} : # of individuals of type i in the nth generation

$\{\boldsymbol{Z}_n\}_{n\geq 0}$: ∞-dim Markov process with abs. state $\mathbf{0} = (0, 0, \ldots)$.

Multi-type Galton-Watson process

Progeny generating vector $\boldsymbol{G}(\boldsymbol{s}) = (G_1(\boldsymbol{s}), G_2(\boldsymbol{s}), G_3(\boldsymbol{s}), \ldots)$, where $G_i(\boldsymbol{s})$ is the progeny generating function of an individual of type i

$$G_i(\boldsymbol{s}) = \mathbb{E}\left(\boldsymbol{s}^{\boldsymbol{Z}_1} \middle| \varphi_0 = i\right) = \sum_{\boldsymbol{r}} p_i(\boldsymbol{r}) \prod_{k=1}^{\infty} s_k^{r_k}, \qquad \boldsymbol{s} \in [0,1]^{\mathcal{X}}.$$

Mean progeny matrix M with elements

$$m_{ij} = \left.\frac{\partial G_i(\boldsymbol{s})}{\partial s_j}\right|_{\boldsymbol{s}=\boldsymbol{1}}$$

$=$ expected number of direct offspring of type j born to a parent of type i

Extinction probabilities

For $A \subseteq \mathcal{X}$ the extinction probability vector $\boldsymbol{q}(A)$ has entries

$$q_i(A) = \mathbb{P}\left[\lim_{n\to\infty} \sum_{\ell\in A} Z_{n\ell} = 0 \,\middle|\, \varphi_0 = i\right]$$

Global extinction probability vector : ext. of the whole process

$$\boldsymbol{q} = \boldsymbol{q}(\mathcal{X})$$

Partial extinction probability vector : ext. of all types

$$\tilde{\boldsymbol{q}} = \lim_{k\to\infty} \boldsymbol{q}(\{1, \ldots, k\})$$

We have

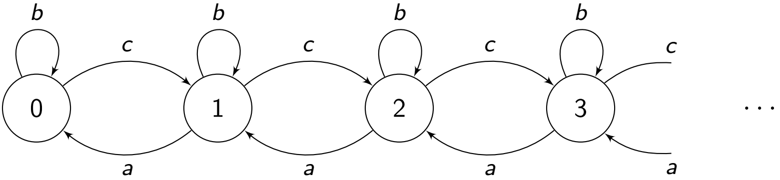
$$\mathbf{0} \leq \boldsymbol{q} \leq \tilde{\boldsymbol{q}} \leq \mathbf{1}$$

Example : nearest neighbour BRW

Suppose $G_0(\boldsymbol{s}) = G_0(s_0, s_1)$, and $G_i(\boldsymbol{s}) = G_i(s_{i-1}, s_i, s_{i+1}), i \geq 1$ with mean progeny matrix

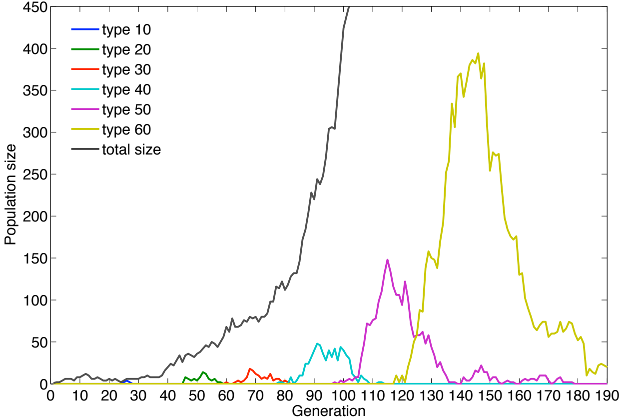
$$M = \begin{bmatrix} b & c & 0 & 0 & 0 & \cdots \\ a & b & c & 0 & 0 & \\ 0 & a & b & c & 0 & \\ 0 & 0 & a & b & c & \\ \vdots & & & \ddots & \ddots & \ddots \end{bmatrix},$$

which can be represented as



Example : nearest neighbour BRW

$a = 1/20,\ b = 1/2,\ c = 1/2\,;$
$b + 2\sqrt{ac} < 1,\ a + b + c > 1,\ c > a$



In this case $\boldsymbol{q} < \widetilde{\boldsymbol{q}} = \mathbf{1}$.

Fixed points

For any $A \subseteq \mathcal{X}$ the vector $\boldsymbol{q}(A)$ satisfies the fixed point equation

$$\boldsymbol{s} = \boldsymbol{G}(\boldsymbol{s}).$$

That is, $\boldsymbol{q}(A)$ is an element of

$$S = \{\boldsymbol{s} \in [0,1]^\infty : \boldsymbol{s} = \boldsymbol{G}(\boldsymbol{s})\}.$$

The set S of fixed points in the irreducible case

The vector $\boldsymbol{q}$ is the minimal non-negative element of S

Finite type case :

- The set S contains at most two elements, $\boldsymbol{q} = \widetilde{\boldsymbol{q}}$ and $\mathbf{1}$.

Infinite type case :

- Moyal (1962) : S contains at most a single solution with $\limsup_i s_i < 1$ (corresponding to $\boldsymbol{q}$).
- Spataru (1989) : S contains at most two elements, $\boldsymbol{q}$ and $\mathbf{1}$.
- Bertacchi and Zucca (2014,2015) : proved the inaccuracy of the later and provided an irreducible process where S contains uncountably many elements.

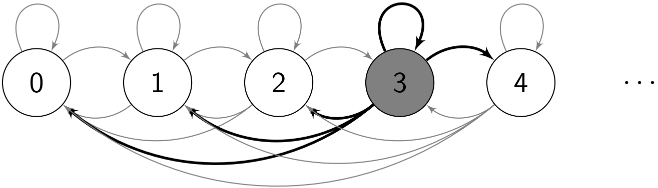
Can we tell more about S and the location of $\boldsymbol{q}(A) \neq \boldsymbol{q}$?

Lower Hessenberg branching processes

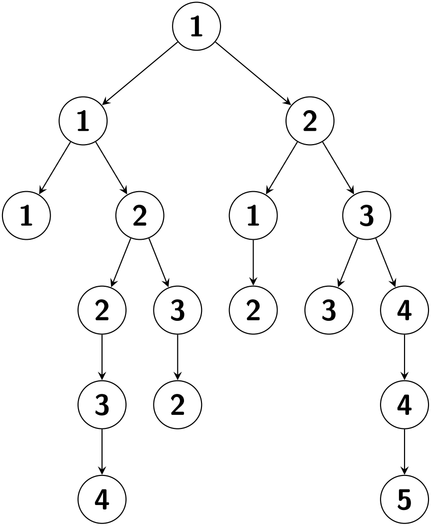
- We assume M is lower Hessenberg

$$M = \begin{bmatrix} m_{00} & m_{01} & 0 & 0 & 0 & \dots \\ m_{10} & m_{11} & m_{12} & 0 & 0 & \\ m_{20} & m_{21} & m_{22} & m_{23} & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}$$

- Type $i \geq 0$ individuals cannot have offspring of type $j > i+1$.
- We assume $m_{i,i+1} > 0$ for all $i \geq 0$.

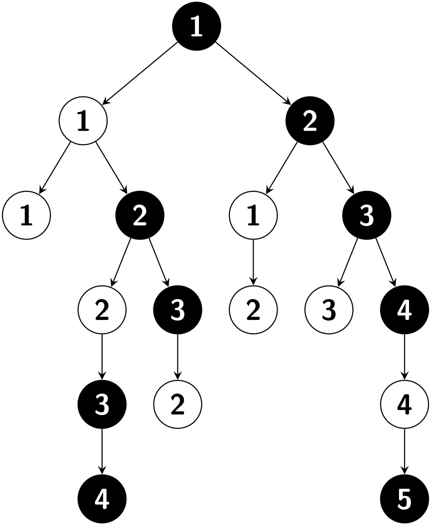


Embedded Galton-Watson process in varying environment



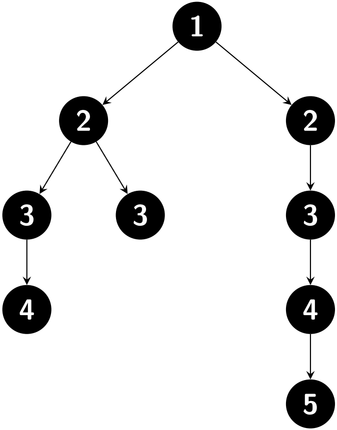
$\{Z_n\}$

Embedded Galton-Watson process in varying environment



$\{Z_n\} \to \{Y_k\}$

Embedded Galton-Watson process in varying environment



$\{Y_k\}$

Embedded Galton-Watson process in varying environment

$\{Y_k\}$ has two absorbing states, 0 and ∞.

Theorem (Braunsteins and H., 2017)

$$\textit{Partial extinction in } \{\boldsymbol{Z}_n\} \overset{a.s}{\Longleftrightarrow} Y_k < \infty \quad \textit{for all } k \geq 0$$

$$\textit{Global extinction in } \{\boldsymbol{Z}_n\} \overset{a.s}{\Longleftrightarrow} Y_k = 0 \quad \textit{for some } k \geq 0$$

The progeny generating functions $g_k(s) = E[s^{Y_{k+1}} | Y_k = 1]$ may be defective, that is, $g_k(1) \leq 1$.

We derived implicit and explicit expressions for $g_k(s)$ in terms of $\boldsymbol{G}(\boldsymbol{s})$, and recursive expressions for the first two moments $\mu_k = g_k'(1)$ and $a_k = g_k''(1)$.

Extinction Criteria

Theorem (Braunsteins and H., 2017)

Suppose

$$\mu_0 = \frac{m_{01}}{1 - m_{00}} \quad \text{and} \quad \mu_k = \frac{m_{k,k+1}}{1 - \sum_{i=1}^{k} m_{ki} \prod_{j=i}^{k-1} \mu_j},$$

then

$$\widetilde{\boldsymbol{q}} = \mathbf{1} \quad \Leftrightarrow \quad 0 \leq \mu_k < \infty \ \forall\, k \geq 0$$

and, when $\widetilde{\boldsymbol{q}} = \mathbf{1}$,

$$\boldsymbol{q} = \mathbf{1} \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} \left(\prod_{\ell=1}^{j} \mu_\ell \right)^{-1} = \infty.^{*}$$

* : under some second moment assumptions.

Fixed points

Recall that $S = \{\boldsymbol{s} \in [0,1]^\infty : \boldsymbol{s} = \boldsymbol{G}(\boldsymbol{s})\}$.

Here,

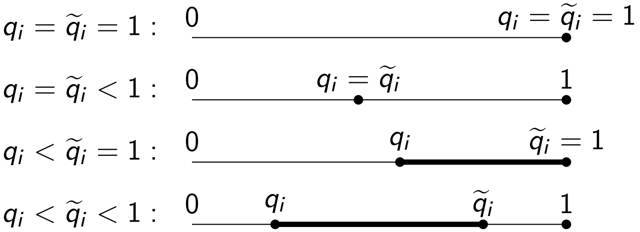
$$s_i = G_i(s_0, s_1, \dots, s_i, s_{i+1}), \quad i \geq 0.$$

$\rightarrow$ It suffices to study the one-dimensional projection sets :

$$S_i = \{x \in [0,1] : \exists\, \boldsymbol{s} \in S, \text{such that } s_i = x\}.$$

Fixed points

Illustration of S_i :



Theorem (Braunsteins and H., 2017)

Suppose $\{\boldsymbol{Z}_n\}$ *is irreducible. If* $S = \{\mathbf{1}\}$ *then* $\boldsymbol{q} = \widetilde{\boldsymbol{q}} = \mathbf{1}$, *otherwise*

$$\boldsymbol{q} = \min S \quad \textit{and} \quad \widetilde{\boldsymbol{q}} = \sup S \backslash \{\mathbf{1}\}.$$

In particular,

$$S_i = [q_i, \widetilde{q}_i] \cup 1, \quad i \geq 0.$$

General extinction events

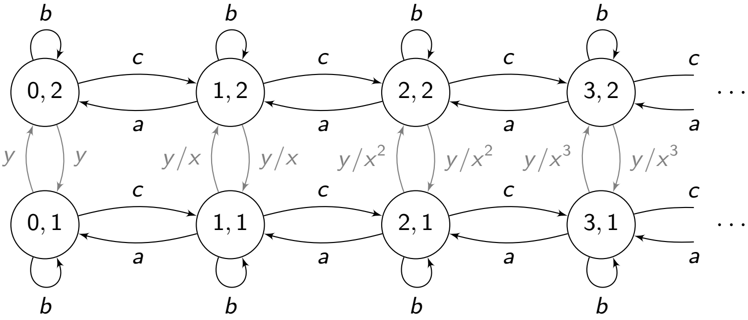
In an irreducible lower Hessenberg branching process, $\boldsymbol{q}(A)$ takes at most two distinct values :

- $\boldsymbol{q}(A) = \widetilde{\boldsymbol{q}}$ if $|A| < \infty$
- $\boldsymbol{q}(A) = \boldsymbol{q}$ if $|A| = \infty$

$\rightarrow$ for LHBPs, we have identified the location of all $\boldsymbol{q}(A)$ in S.

Now we add layers...

Example : the double nearest-neighbour BRW



Depending on the parameter values, $\boldsymbol{q}(A)$ takes one of up to four different values.

Block lower Hessenberg branching processes

- With $d \geq 1$ layers, the type space is $\mathcal{X}_d = \mathcal{X} \times \{1, \ldots, d\}$.
- The mean progeny matrix M is block lower Hessenberg

$$M = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 & 0 & \ldots \\ M_{21} & M_{22} & M_{23} & 0 & 0 & \\ M_{31} & M_{32} & M_{33} & M_{34} & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}$$

- Each M_{ij} is an $d \times d$ matrix
- We assume M is irreducible

Extinction in layers

We still have

- $\boldsymbol{q}(A) = \widetilde{\boldsymbol{q}}$ if $|A| < \infty$
- $\boldsymbol{q}(\mathcal{X}_d) = \boldsymbol{q}$.

However, now we can have $\boldsymbol{q}(A) > \boldsymbol{q}$ for some $|A| = \infty$.

A_i = the infinite set of types forming the ith layer, $1 \leq i \leq d$.

A process can now survive in A_2 while enduring extinction in A_1.

We consider sets $A \in \sigma(A_1, \ldots, A_d)$ and their complement $\bar{A}$.

When do we have $\boldsymbol{q} < \boldsymbol{q}(A) < \widetilde{\boldsymbol{q}}$?

Extinction in layers

When do we have $\boldsymbol{q} < \boldsymbol{q}(A) < \widetilde{\boldsymbol{q}}$?

Theorem (Braunsteins and H., 2018)

Let $A \in \sigma(A_1, \dots, A_d)$, *and assume* $\tilde{\boldsymbol{q}}^{(\bar{A})} < \mathbf{1}$ *and* $\nu(\tilde{M}^{(\bar{A})}) < 1$. *If, in addition,*

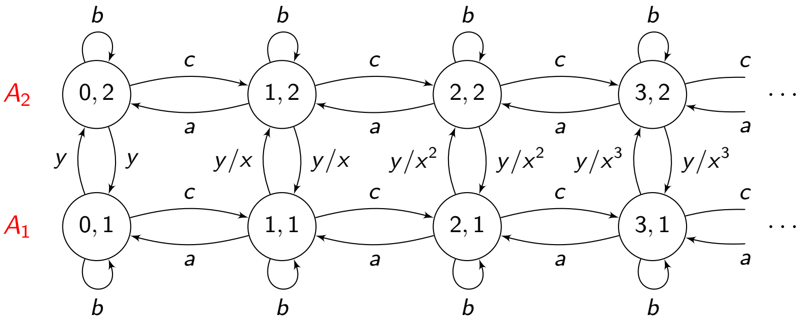
(A) $\sum_{k=0}^{\infty}(\mathbf{1}_v^\top \boldsymbol{t}_k^{(\bar{A})})\, \tilde{\mathcal{M}}_{0\to k-1}^{(\bar{A})} \mathbf{1}_v < \infty$, *and*

(B) *there exists* $K < \infty$ *such that* $\tilde{F}_k^{(\bar{A})} \leq K\, \mathbf{1}_v \cdot \mathbf{1}_v^\top$ *for all* $k \geq 0$,

then $\boldsymbol{q} < \boldsymbol{q}(A)$ *and* $\boldsymbol{q}(\bar{A}) < \tilde{\boldsymbol{q}}$.

Example : double nearest-neighbour BRW

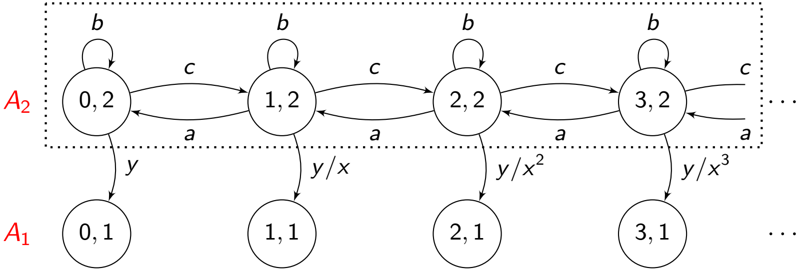
$A = A_1, \ \bar{A} = A_2$



Example : double nearest-neighbour BRW

A_2 is able to globally survive without the help of A_1 but becomes partially extinct,

(A) + (B) : finite expected number of (sterile) types in A_1 from A_2



$\rightarrow \boldsymbol{q} < \boldsymbol{q}(A_1)$ and $\boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}}$

$\rightarrow \boldsymbol{q} < \boldsymbol{q}(A_1), \boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}}$

Example : double nearest-neighbour BRW

Proposition (Braunsteins and H., 2018)

Suppose $b + 2\sqrt{ac} < 1$ *and*

$$\mu := \left(1 - b - \sqrt{(1-b)^2 - 4ac}\right)/2a > 1.$$

We have

(i) *if* $x = 1$ *and* $b + y + 2\sqrt{ac} \leq 1$, *then* $\boldsymbol{q} = \boldsymbol{q}(A_1) = \boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}} = \mathbf{1}$;

(ii) *if* $x = 1$ *and* $b + y + 2\sqrt{ac} > 1$, *then* $\boldsymbol{q} = \boldsymbol{q}(A_1) = \boldsymbol{q}(A_2) = \widetilde{\boldsymbol{q}} < \mathbf{1}$;

(iii) *if* $x > 1$, *then* $\boldsymbol{q} < \widetilde{\boldsymbol{q}}$;

(iv) *if* $x > \mu$, *then* $\boldsymbol{q} < \boldsymbol{q}(A_1) < \widetilde{\boldsymbol{q}}$ *and* $\boldsymbol{q} < \boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}}$.

Example : double nearest-neighbour BRW

$a = 1/5, b = 0, c = 1, y = 1/5 \to \mu = 1.38, b + y + 2\sqrt{ac} = 1.09$

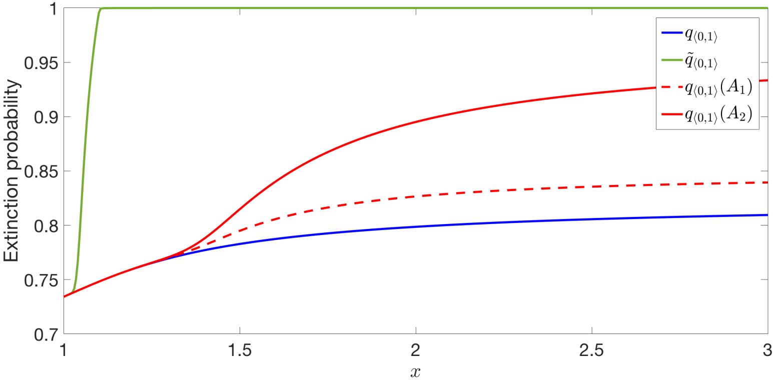
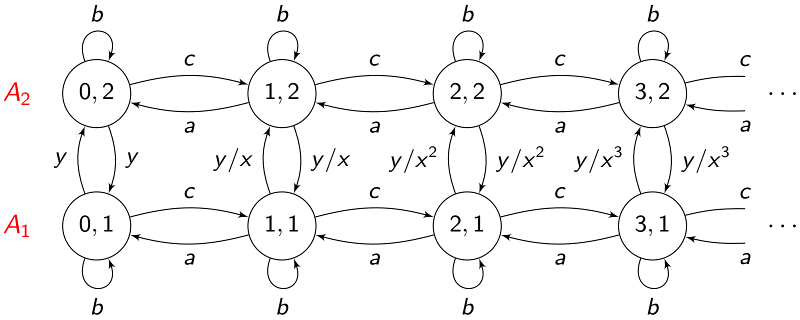


FIGURE – The extinction probabilities $q_{\langle 0,1\rangle}$, $q_{\langle 0,1\rangle}(A_1)$, $q_{\langle 0,1\rangle}(A_2)$ and $\tilde{q}_{\langle 0,1\rangle}$ for $1 \leq x \leq 3$.

Example : double nearest-neighbour BRW

We study the set of fixed points S by projecting it on types 0

$\rightarrow$ 2-d projection set S_0



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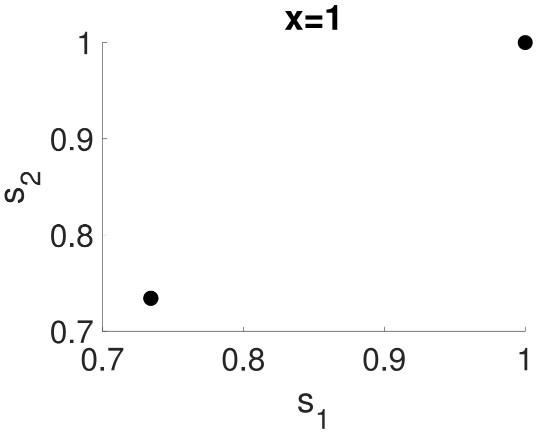
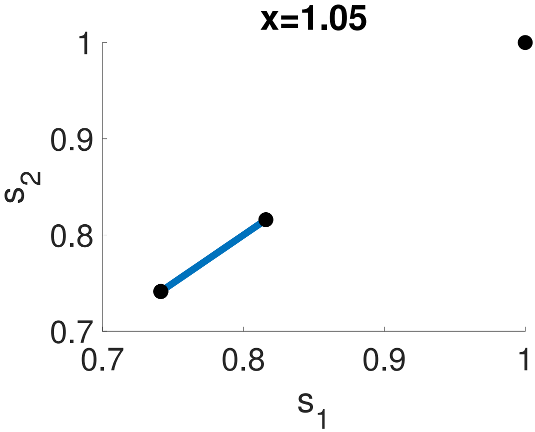


FIGURE – The projection set S_0 for a specific value of x (with the shorthand notation s_i for $s_{\langle 0,i \rangle}$, $i = 1, 2$).

Example : double nearest-neighbour BRW



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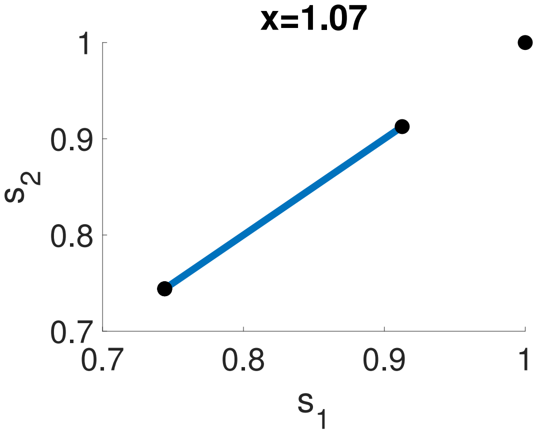

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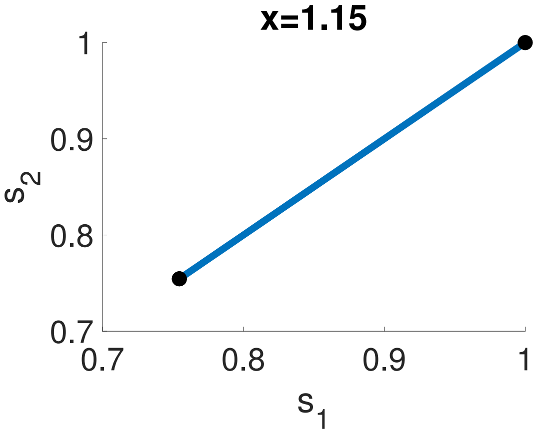

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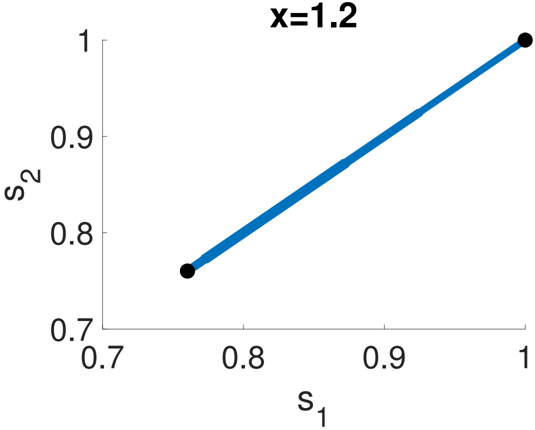

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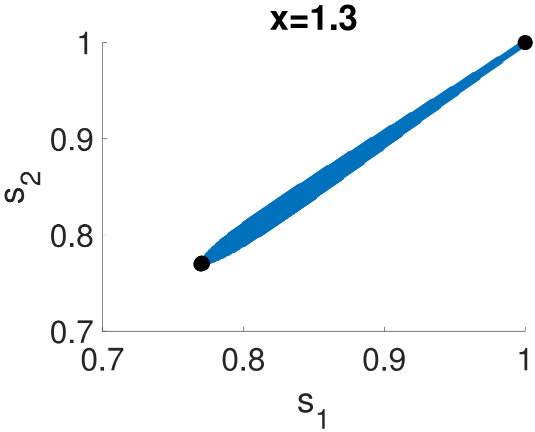

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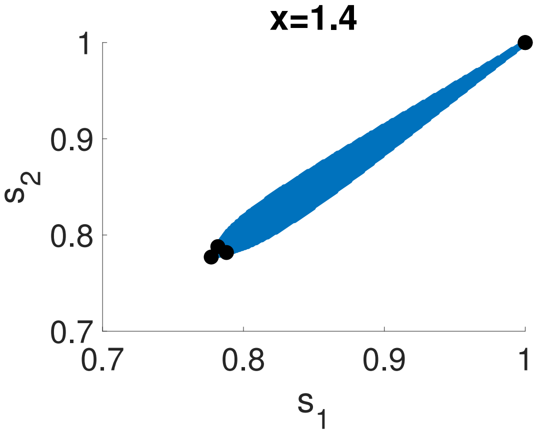

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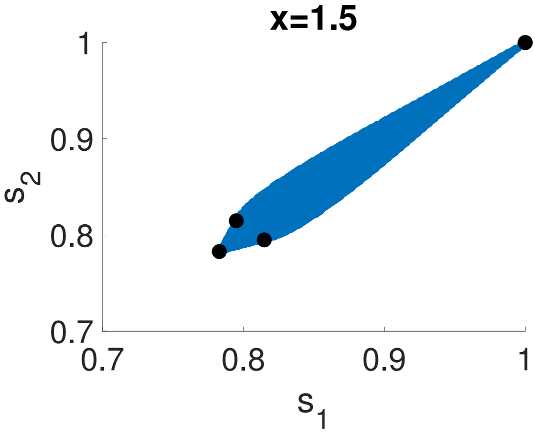

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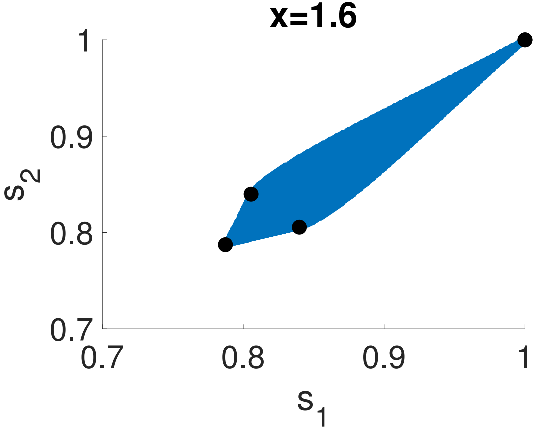
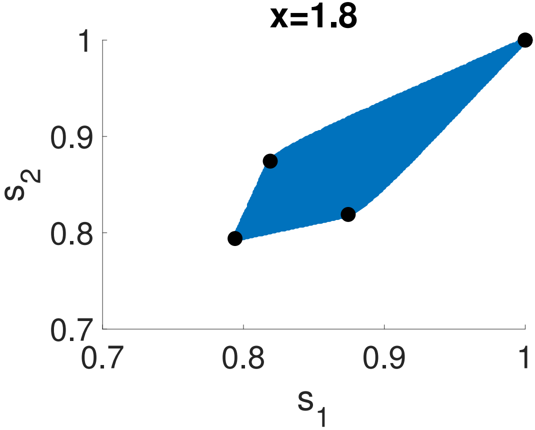


FIGURE – The projection set S_0 for $y = 1/5$ and a specific value of x (with the shorthand notation s_i for $s_{\langle 0,i \rangle}$, $i = 1, 2$).

Example : double nearest-neighbour BRW



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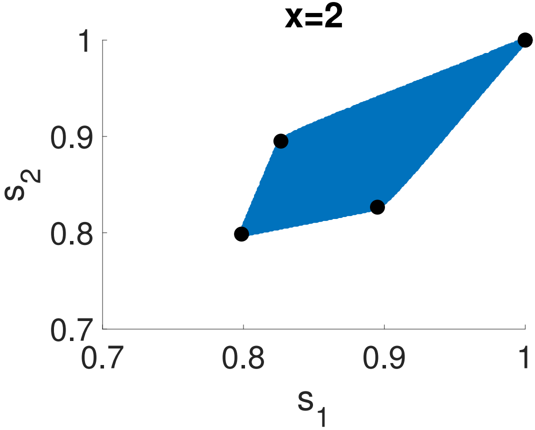

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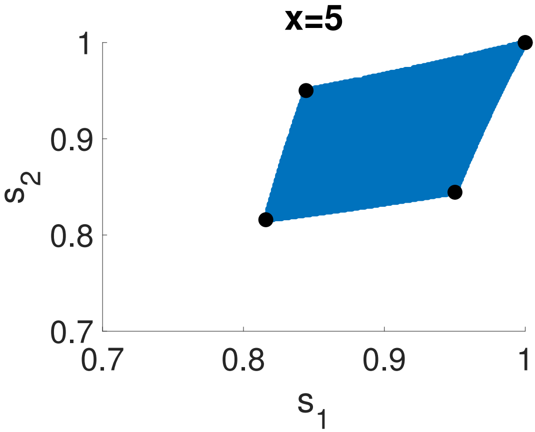

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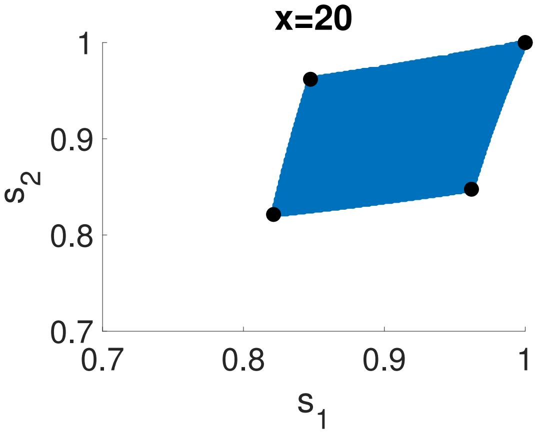

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Conjecture

If $\boldsymbol{q} = \widetilde{\boldsymbol{q}}$ then $S = \{\boldsymbol{q}, \mathbf{1}\}$, whereas if $\boldsymbol{q} < \widetilde{\boldsymbol{q}}$ then S contains a continuum of elements, whose minimum is $\boldsymbol{q}$, and whose maximum is $\widetilde{\boldsymbol{q}}$.

In addition, the boundary of any projection set is differentiable everywhere except at each point that corresponds to an extinction probability vector $\boldsymbol{q}(A)$ for some $A \subseteq \mathcal{X}_d$.

We believe that this conjecture applies more generally to *any* irreducible branching process with countably many types.

References

The material of this talk is in

P. Braunsteins and S. Hautphenne
Extinction in lower Hessenberg branching processes with countably many types.
arXiv preprint arXiv :1706.02919, 2017.

P. Braunsteins and S. Hautphenne
The probabilities of extinction in a branching random walk on a strip, 2018. Soon on arXiv.