Continuous-State Branching Processes with Competition
Duality and Reflection at Infinity

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Introduction

Consider a random continuous population with heuristically the following dynamics:

- Each individual reproduces independently from the others, with a same law (as in a Continuous-State Branching Process)
- At constant rate c, a "pair" of individuals is picked at random and one kills the other (quadratic competition).

The total size of the population is called **logistic CSBP** (Lambert AAP 05). In the case without jumps, the process is the logistic Feller diffusion :

$$\mathrm{d}Z_t = \sigma\sqrt{Z_t}\mathrm{d}B_t + \gamma Z_t \mathrm{d}t - \frac{c}{2}Z_t^2 \mathrm{d}t. \quad (1)$$

Aim: study these processes with *a general branching mechanism* Ψ and classify the boundaries

CSBPs without competition

A general branching mechanism takes the form

$$\Psi(z) = -\lambda + \frac{\sigma^2}{2} z^2 + \gamma z + \int_0^{+\infty} \left(e^{-zx} - 1 + zx \mathbb{1}_{\{x \leq 1\}}\right) \pi(\mathrm{d}x)$$

where

- π: a measure over $(0, \infty)$ s.t. $\int_0^{\infty} (1 \wedge x^2) \pi(\mathrm{d}x) < \infty$
- $\lambda \geq 0$: a killing rate, understood as a jump to ∞ at rate λz
- $\gamma \in \mathbb{R}$: a deterministic drift, $\sigma \geq 0$, the Feller diffusion part

Let $(Z_t, t \geq 0)$ be a Ψ-CSBP

$$\mathbb{E}_z[e^{-xZ_t}] = e^{-zu_t(x)}, \text{ with } \frac{\mathrm{d}u_t(x)}{\mathrm{d}t} = -\Psi(u_t(x))$$

- It explodes (reaches ∞) with positive probability iff $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$.
- It reaches 0 with positive probability iff $\int^{\infty} \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$.

Questions

(1) *Are there strong enough reproduction laws to face the competition and explosion to occur* (∞ **accessible**)?

(2) *If the process does not explode, is it possible to start it from infinity ?* (∞ **entrance**.)

(3) *If the process explodes, is the competition strong enough to push back the process in* $[0, \infty)$ *or not ?* (∞ **regular reflecting** *or* **exit**.)

(4) *What are the possible long-term behaviors? Is there a stationary law?*

...

Reflecting means that $\lambda(\{t > 0; Z_t = \infty\}) = 0$.

Minimal Logistic CSBPs: definition

Denote by $\mathcal{G}$ the extended generator of a CSBP and set

$$\mathcal{L}f(z) := \mathcal{G}f(z) - \frac{c}{2}z^2 f'(z).$$

Definition

A **minimal** *logistic continuous-state branching process is a càdlàg Markov process* $(Z_t^{min}, t \geq 0)$ *on* $[0, \infty]$ *with* 0 *and* ∞ **absorbing**, *satisfying* **(MP)**. *For any function* $f \in C_c^2((0,\infty))$, *the process*

$$t \in [0, \zeta) \mapsto f\left(Z_t^{min}\right) - \int_0^t \mathcal{L}f\left(Z_s^{min}\right) \mathrm{d}s \qquad \textbf{(MP)}$$

is a martingale under each $\mathbb{P}_z$, *with* $\zeta := \inf\{t \geq 0; Z_t^{min} \notin (0,\infty)\}$.

By minimal process, we mean that if it explodes, the process remains at ∞ from its explosion time $\zeta_\infty := \inf\{t \geq 0, Z_t^{\min} = \infty\}$.

Minimal Logistic CSBP

Theorem

There exists a unique minimal logistic CSBP.

Theorem (Accessibility of ∞)

Assume $c > 0$. The boundary ∞ is inaccessible for $(Z_t^{min}, t \geq 0)$ if and only if

$$\mathcal{E} := \int_0^{\theta} \frac{1}{x} \exp\left(\frac{2}{c}\int_x^{\theta} \frac{\Psi(u)}{u}\,\mathrm{d}u\right)\mathrm{d}x = \infty,$$

for some arbitrary $\theta > 0$.

Construction of minimal logistic CSBPs

Consider $(Y_t, t \geq 0)$ a sp-Lévy process with Laplace exponent $-\Psi$, killed at ∞ at an independent exponential r.v. $\mathbb{e}_\lambda$ with parameter $\lambda := -\Psi(0) \geq 0$. Set $(R_t, t \geq 0)$ the generalized Ornstein-Uhlenbeck process defined by

$$R_t = z + Y_t - \frac{c}{2}\int_0^t R_s \mathrm{d}s.$$

Set $\sigma_0 := \inf\{t \geq 0, R_t < 0\}$, $\theta_t := \int_0^{t\wedge\sigma_0} \frac{\mathrm{d}s}{R_s}$ and its right-inverse $t \mapsto C_t := \inf\{u \geq 0; \theta_u > t\} \in [0, \infty]$. Let

$$Z_t^{\min} = \begin{cases} R_{C_t} & 0 \leq t < \theta_\infty \\ 0 & t \geq \theta_\infty \text{ and } \sigma_0 < \infty \\ \infty & t \geq \theta_\infty \text{ and } \sigma_0 = \infty. \end{cases}$$

$(Z_t^{\min}, t \geq 0)$ is a minimal logistic continuous-state branching process.

Explosion criterion

The process $(Z_t^{\min}, t \geq 0)$ hit ∞ if and only if $\sigma_0 = \infty$ and

$$\theta_\infty = \int_0^\infty \frac{\mathrm{d}s}{R_s} < \infty.$$

Shiga (PTRF 90) shows that $(R_s, s \geq 0)$ is recurrent if $\mathcal{E} = \infty$ and transient if $\mathcal{E} < \infty$:

- if $(R_s, s \geq 0)$ is recurrent then $\int_0^\infty \frac{\mathrm{d}s}{R_s} = \infty$ on $\sigma_0 = \infty$.
- if $(R_s, s \geq 0)$ is transient, one can show that

$$\mathbb{E}_z \left[\int_0^\infty \frac{\mathrm{d}s}{R_s} ; \sigma_0 = \infty \right] < \infty$$

from

$$\mathbb{E}_z(e^{-\theta R_s}) = \exp\left(-\theta e^{-\frac{c}{2}s} z + \int_0^s \Psi(e^{-\frac{c}{2}u}\theta)\mathrm{d}u \right).$$

Duality of generators

For all $x \in [0, \infty[$ and $z \in [0, \infty[$, let $e_x(z) := e^{-xz} = e_z(x)$, then

Lemma (Generator duality)

$$\mathcal{L}e_x(z) = \mathcal{A}e_z(x) \textit{ with } \mathcal{A}f(x) = \frac{c}{2}xf''(x) - \Psi(x)f'(x).$$

Proof.

$$\mathcal{L}e_x(z) = \Psi(x)ze_x(z) + \frac{c}{2}xz^2e_x(z) = -\Psi(x)\frac{\partial e_z(x)}{\partial x} + \frac{c}{2}x\frac{\partial^2 e_z(x)}{\partial x^2}. \qquad \square$$

There exists a unique strong solution to

$$\mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t - \Psi(U_t)\mathrm{d}t \quad (\star),$$

up to $\tau := \inf\{t > 0, U_t \notin (0, \infty)\}$ (Ψ is locally lipschitz on $(0, \infty)$). However, 0 can be exit, regular or entrance and **there is not a unique semi-group** associated to $\mathcal{A}$.

In the sequel, we say that a process $(Z_t, t \geq 0)$ extends the minimal process if $(Z_t, t \geq 0)$ takes its values in $[0, \infty]$ and $(Z_{t\wedge\zeta_\infty}, t \geq 0) \overset{\mathcal{L}}{=} (Z_t^{\min}, t \geq 0)$ under $\mathbb{P}_z$ for any $z \in [0, \infty)$.

The boundaries behaviors can be summarized as follows

Condition	Boundary of U	Boundary of Z
$\mathcal{E} = \infty$	0 exit	∞ entrance
$\mathcal{E} < \infty$, $0 \leq 2\lambda/c < 1$	0 regular (absorbing)	∞ regular (reflecting)
$2\lambda/c \geq 1$	0 entrance	∞ exit
$\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} < \infty$	∞ entrance	0 exit
$\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} = \infty$	∞ natural	0 natural

Table: Boundaries of Z and boundaries of U

$\rightarrow$ *A duality relation for entrance and exit laws for Markov processes, Cox, Rösler: SPA 84*

Infinity as an entrance boundary: $\mathcal{E} = \infty$

- With $\mathcal{L}e_x(z) = \mathcal{A}e_z(x)$, and $\mathcal{E} = \infty$, a duality result of Ethier and Kurtz yields:

$$\mathbb{E}_z[e^{-xZ_t^{\min}}] = \mathbb{E}_x[e^{-zU_t}], \; z \in [0, \infty), x \in (0, \infty).$$

- Set $P_t e_x(z) := \mathbb{E}_z[e^{-xZ_t^{\min}}]$ for $z \in [0, \infty[$. When $\mathcal{E} = \infty$, since 0 is an **exit** of U then

$$P_t e_x(\infty) := \lim_{z \to \infty} \mathbb{E}_z[e^{-xZ_t^{\min}}] = \mathbb{P}_x(U_t = 0) = \mathbb{P}_x(\tau_0 \leq t) > 0.$$

- One can check that $P_t C_b \subset C_b$, $x \mapsto \mathbb{P}_x(\tau_0 \leq t)$ is the Laplace transform of an **entrance law** and $(P_t, t \geq 0)$ is a Feller semigroup.

Theorem (Infinity as entrance boundary)

The process $(Z_t, t \geq 0)$ such that for all $t \geq 0$, all $z \in [0, \infty]$ and $x \in [0, \infty)$ $\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t})$ is Feller and has ∞ as entrance boundary

Example

Consider $\alpha \in (0,2]$, $\alpha \neq 1$ and $\Psi(z) = (\alpha - 1)z^{\alpha}$, then $\mathcal{E} = \infty$ and ∞ is an **entrance** boundary. For any $t \geq 0$, $z \in [0,\infty]$ and $x \in [0,\infty[$

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t}) \text{ with } \mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t + (1-\alpha)U_t^{\alpha}\mathrm{d}t,$$

the boundary 0 of $(U_t, t \geq 0)$ is an **exit**. Note that when $\alpha \in (0,1)$, the CSBP without competition explodes, so that here competition prevents explosion.

Infinity as regular reflecting boundary: $\mathcal{E} < \infty$ and $0 \leq \frac{2\lambda}{c} < 1$

Given Ψ and $k \geq 1$, define $\pi_k = \pi_{|]0,k[} + (\bar{\pi}(k) + \lambda)\delta_k$ and a branching mechanism Ψ_k by

$$\Psi_k(z) := \frac{\sigma^2}{2} z^2 + \gamma z + \int_0^\infty \left(e^{-zx} - 1 + zx\mathbb{1}_{x \in (0,1)}\right) \pi_k(\mathrm{d}x).$$

Call $(Z_t^{(k)}, t \geq 0)$ the càdlàg logistic CSBP with mechanism Ψ_k and ∞ as entrance boundary.

Theorem (Infinity as regular reflecting boundary)

Assume $\mathcal{E} < \infty$ *and* $0 \leq \frac{2\lambda}{c} < 1$ $(Z_t^{(k)}, t \geq 0) \Longrightarrow (Z_t, t \geq 0)$ *a Feller process extending* $(Z_t^{min}, t \geq 0)$*, with* ∞ **regular reflecting**, *such that for all* $t \geq 0$*, all* $z \in [0, \infty]$ *and* $x \in [0, \infty)$*,*

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t^0})$$

where $(U_t^0, t \geq 0)$ *is solution to* $(\star)$ *with* 0 **regular absorbing**.

Infinity as exit boundary: $\frac{2\lambda}{c} \geq 1$

Theorem (Infinity as exit boundary)

Assume $\frac{2\lambda}{c} \geq 1$ *then* 0 *is an* **entrance** *for* $(U_t, t \geq 0)$, $(Z_t^{(k)}, t \geq 0) \Longrightarrow (Z_t, t \geq 0)$ *a Feller process, extending* $(Z_t^{min}, t \geq 0)$*, with* ∞ **exit** *and for all* $t \geq 0$*, all* $z \in [0,\infty]$ *and* $x \in (0,\infty)$,

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t}).$$

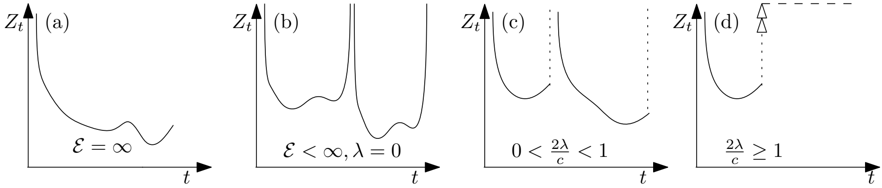

Figure: Symbolic representation of the four behaviors at ∞.

Proposition

If $\mathcal{E} < \infty$ *and* $0 \leq \frac{2\lambda}{c} < 1$ *then* ∞ *is regular for itself, that is* $S_\infty := \inf\{t > 0, Z_t = \infty\}$ *is such that* $\mathbb{P}_\infty(S_\infty = 0) = 1$.

In particular, there are infinitely many small excursions from ∞, and a local time at ∞.

Example (Squared Bessel processes)

Let $\lambda > 0$ and $\pi \equiv 0$ in order that $\Psi(x) = -\lambda$ for all $x \geq 0$.

- If $\frac{2\lambda}{c} < 1$ then ∞ is **regular reflecting** and $\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t^0})$ with $\mathrm{d}U_t^0 = \sqrt{cU_t^0}\mathrm{d}B_t + \lambda \mathrm{d}t$ and 0 **regular absorbing**.
- If $\frac{2\lambda}{c} \geq 1$ then ∞ is an **exit** and $\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t})$ with $\mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t + \lambda \mathrm{d}t$, and 0 is an **entrance**.

$\rightarrow$*Fast-fragmentation-coalescence process, Kyprianou et al. AoP17*

Example with **continuous explosion** and a phase transition between entrance and regular.

Example

Consider $\alpha > 0$, $\beta > 0$ and set $\pi(\mathrm{d}u) = \frac{\alpha}{u(\log u)^{\beta+1}} \mathbb{1}_{\{u \geq 2\}} \mathrm{d}u$.

i) If $\beta = 1$ and $\frac{2\alpha}{c} \leq 1/2$ then $\mathcal{E} = \infty$ and ∞ is an **entrance** boundary.
ii) If $\beta = 1$ and $\frac{2\alpha}{c} > 1/2$ then $\mathcal{E} < \infty$ and ∞ is a **regular reflecting** boundary.
iii) If $\beta \in]0, 1[$, then $\mathcal{E} < \infty$ and ∞ is a **regular reflecting** boundary.

Infinity as reflecting or exit boundary: sketch of proof.

Assume $\mathcal{E} < \infty$. Set $(U_t^{(k)}, t \geq 0)$ the Ψ_k-generalized Feller diffusion (with 0 exit):

$$\mathbb{E}_z[e^{-xZ_t^{(k)}}] = \mathbb{E}_x[e^{-zU_t^{(k)}}].$$

- For all x, $\Psi_{k+1}(x) \leq \Psi_k(x)$ so by the comparison theorem: $U_t^{(k+1)} \geq U_t^{(k)}$ for all t a.s. Thus a.s. for all t, $U_t^{(k)} \to U_t^{(\infty)}$.
- $||\mathcal{A}^{(k)}f - \mathcal{A}f||_\infty \to 0$ for any $f \in C_c^2$. Thus $(U_t^{(\infty)}, t \leq \tau^\infty)$ with $\tau^\infty := \inf\{t; U_t^{(\infty)} = 0\}$, has the same law as the minimal diffusion with generator $\mathcal{A}$ and $\mathbb{P}_x(\tau^\infty < \infty) > 0$ iff $\frac{2\lambda}{c} < 1$.
- $U_{t+\tau^\infty}^{(\infty)} = \lim U_{t+\tau^\infty}^{(k)} = 0$ since $\tau^\infty \geq \tau^{(k)}$ and 0 is an exit of $(U_t^{(k)}, t \geq 0)$. Then $(U_t^{(\infty)}, t \geq 0)$ has 0 regular absorbing if $\frac{2\lambda}{c} < 1$ or entrance if $\frac{2\lambda}{c} \geq 1$.

Conclusion: if $\frac{2\lambda}{c} < 1$ then $\mathbb{E}_x[e^{-zU_t^{(k)}}] \underset{k\to\infty}{\longrightarrow} \mathbb{E}_x[e^{-zU_t^0}]$.

Let $(P_t^{(k)}, t \geq 0)$ the semi-group of $(Z_t^{(k)}, t \geq 0)$. Set

$$P_t e_x(z) := \lim_{k\to\infty} P_t^{(k)} e_x(z) = \mathbb{E}_x[e^{-zU_t^{(\infty)}}].$$

By Stone-Weierstrass: $P_t C_b \subset C_b$. One has

$$||P_t^{(k)} e_x - P_t e_x||_\infty = \sup_{z\in[0,\infty]} \left(\mathbb{E}_x[e^{-zU_t^{(k)}}] - \mathbb{E}_x[e^{-zU_t^{(\infty)}}]\right) \underset{k\to\infty}{\longrightarrow} 0$$

By Stone-Weierstrass: $||P_t^{(k)} f - P_t f||_\infty \longrightarrow 0$ for any $f \in C_b$. We deduce from this, that:

- $(P_t, t \geq 0)$ is a semigroup with the Feller property.
- Ethier-Kurtz (Thm 2.5 p167): $(Z_t^{(k)}, t \geq 0) \Longrightarrow (Z_t, t \geq 0)$

Set $(Z_t, t \geq 0)$ the Markov process on $[0, \infty]$ with semigroup $(P_t, t \geq 0)$. One has

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t^{(\infty)}}].$$

It remains to show that $(Z_t, t \geq 0)$ is an extension of $(Z_t^{\min}, t \geq 0)$.

For any $f \in C_c^2$, $||\mathcal{L}^{(k)}f - \mathcal{L}f||_\infty \to 0$. Therefore:

$$\left(f(Z_t) - \int_0^t \mathcal{L}f(Z_s)\mathrm{d}s, t \geq 0\right) \text{ is a martingale.}$$

Stopping at time ζ_∞ yields that

$$\left(f(Z_{t\wedge\zeta_\infty}) - \int_0^t \mathcal{L}f(Z_{s\wedge\zeta_\infty})\mathrm{d}s, t \geq 0\right) \text{ is a martingale}$$

Thus $(Z_{t\wedge\zeta_\infty}, t \geq 0)$ solves **(MP)** and by uniqueness, has the same law as $(Z_t^{\min}, t \geq 0)$.

Conclusion: when $\mathcal{E} < \infty$, ∞ is accessible and

- $\mathbb{E}_\infty[e^{-xZ_t}] = \mathbb{P}_x(\tau_0 \leq t) > 0$ if $\frac{2\lambda}{c} < 1$: ∞ is regular.
- $\mathbb{E}_\infty[e^{-xZ_t}] = \mathbb{P}_x(\tau_0 \leq t) = 0$, if $\frac{2\lambda}{c} \geq 1$, ∞ is an exit.
- If $\frac{2\lambda}{c} < 1$, since 0 is regular absorbing then for any $z \in [0, \infty]$, $\mathbb{P}_z(Z_t < \infty) = \mathbb{E}_{0+}[e^{-zU_t^0}] = 1$ and ∞ is reflecting.

Remarks and open questions

→ Duality can be used to characterize the long-term behavior (extinction for instance), and can be used to compute the Laplace transform of the stationary law when it exists.

? The discrete-state space remains unclear. In this case, the process $(Z_t, t \geq 0)$ corresponds to **the number of fragments in some exchangeable coalescence-fragmentation processes with Kingman coalescence** (Berestycki EJP 04). Accessibility of ∞ is more involved. See Gonzales-Casanova et al. (17+) for a study of a class of discrete branching processes with interactions by duality.

? No information about the local time at ∞ so far.

? Can we use duality for a sticky boundary?

? Can we use duality for studying the logistic process conditioned on the non-extinction?

Thank you for your attention

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- J. T. Cox and U. Rösler, *A duality relation for entrance and exit laws for Markov processes*, SPA (1984).

Corollary (Stationarity)

Assume

$$\Psi(z) = -\lambda - \delta z - \int_0^\infty (1 - e^{-zu})\pi(\mathrm{d}u)$$

with $\delta \geq 0$, $\int_0^\infty (1 \wedge u)\pi(\mathrm{d}u) < \infty$ *and* $0 \leq \frac{2\lambda}{c} < 1$. *Set the condition*

$$\textbf{(A)} : (\delta = 0 \textit{ and } \bar{\pi}(0) + \lambda \leq c/2).$$

- *If* **(A)** *holds then* $(Z_t, t \geq 0)$ *converges in probab. to* 0.
- *If* **(A)** *is not satisfied then* $(Z_t, t \geq 0)$ *converges in law towards the distribution carried over* $(\frac{2\delta}{c}, \infty)$ *whose Laplace transform is*

$$x \in \mathbb{R}_+ \mapsto \mathbb{E}[e^{-xZ_\infty}] := \frac{\int_x^\infty \exp\left(\int_\theta^y \frac{2\Psi(z)}{cz}\mathrm{d}z\right)\mathrm{d}y}{\int_0^\infty \exp\left(\int_\theta^y \frac{2\Psi(z)}{cz}\mathrm{d}z\right)\mathrm{d}y}.$$

Remark

The condition **(A)** *is not satisfied if and only if at least one of the following holds*

$$\lim_{u\to\infty}\frac{\Psi(u)}{u}=-\delta\neq 0,\ \pi((0,1))=\infty,\ \bar{\pi}(0)+\lambda>\frac{c}{2}.$$

This already appears in Lambert 2005 with $\lambda=0$ and a moment assumption.

Proof.

The condition **not (A)** is the NSC for $(U_t^0, t\geq 0)$ to have a positive probability to escape to ∞:

$$\mathbb{E}_z[e^{-xZ_t}]=\mathbb{E}_x[e^{-zU_t^0}]\underset{t\to\infty}{\longrightarrow}\mathbb{P}_x(\tau_0<\tau_\infty)=\frac{s(\infty)-s(x)}{s(\infty)-s(0)},$$

where s is the scale function associated to $\mathcal{A}$. □

Theorem (Long-term behaviors, $0 \leq \frac{2\lambda}{c} < 1$)

Consider $(Z_t, t \geq 0)$ the process started from $z \in (0, \infty)$.

1) *If $0 \leq \frac{2\lambda}{c} < 1$ and $\Psi(z) \geq 0$ for some $z > 0$ then*
   - 1-1) *If $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} = \infty$, then $Z_t > 0$ for any $t \geq 0$ a.s. and $Z_t \underset{t \to \infty}{\longrightarrow} 0$ a.s.*
   - 1-2) *If $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$, then $(Z_t, t \geq 0)$ get absorbed at 0 in finite time almost-surely.*

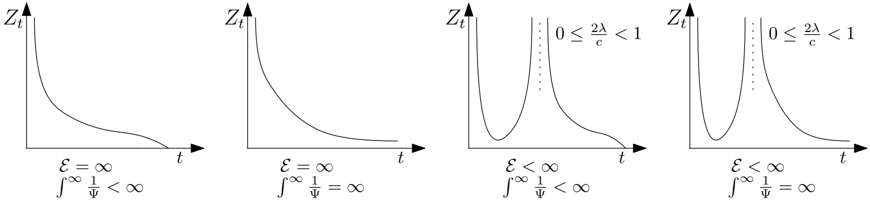

Figure: Representation of the two behaviors at 0 in the non-subordinator case with ∞ entrance or reflecting.

Theorem (Long-term behaviors, $\frac{2\lambda}{c} \geq 1$)

2) *If* $\frac{2\lambda}{c} \geq 1$ *and* $\Psi(z) < 0$ *for all* $z > 0$ *then* $(Z_t, t \geq 0)$ *get absorbed at* ∞ *in finite time almost-surely.*

3) *If* $\frac{2\lambda}{c} \geq 1$ *and* $\Psi(z) \geq 0$ *for some* $z > 0$ *then* $\mathbb{P}_z(Z_t \underset{t\to\infty}{\longrightarrow} 0) = 1 - \mathbb{P}_z(\zeta_\infty < \infty) > 0$ *and* $Z_t > 0$ *for any* $t \geq 0$ *a.s iff* $\int^\infty \frac{\mathrm{d}u}{\Psi(u)} = \infty$.