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# Robust estimation in controlled branching processes: Bayesian estimators via disparities

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# Contents

# Controlled Branching Processes

A Controlled Branching Process is a discrete-time stochastic growth population model in which the individuals with reproductive capacity in each generation are controlled. This branching model is well-suited for describing the probabilistic evolution of populations in which, for various reasons of an environmental, social or other nature, there is a mechanism that establishes the number of progenitors who take part in each generation.

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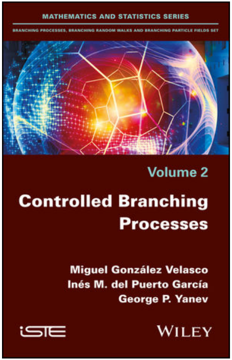

## Definition (Yanev (1975))

Let $\{X_{ni} : n = 0, 1, \ldots; i = 1, 2, \ldots\}$ and $\{\phi_n(k) : n, k = 0, 1, \ldots\}$ be two independent families of non negative integer valued random variables which are defined on the same probability space, $(\Omega, \mathcal{A}, P)$.

(i) $\{X_{ni} : n = 0, 1, \ldots; i = 1, 2, \ldots\}$ are i.i.d. random variables whose distribution is denoted by $p = \{p_k\}_{k\geq 0}$, $p_k = P[X_{01} = k]$, $k \geq 0$.

(ii) For $n = 0, 1, \ldots$, $\{\phi_n(k) : k = 0, 1, \ldots\}$ are independent stochastic processes with equal one-dimensional probability distributions, i.e., for each $n$, $p_j(k) = P[\phi_n(k) = j]$, $j, k \geq 0$.

The stochastic process $\{Z_n\}_{n\geq 0}$ defined as:

$$Z_0 = N \geq 0, \quad Z_{n+1} = \sum_{i=1}^{\phi_n(Z_n)} X_{ni}, \quad n = 0, 1, \quad \left(\sum_1^0 = 0\right),$$

is known as **Controlled Branching Process (CBP) with random control function**.

# Probability model

## Main parameters of the model

- $p = \{p_k\}_{k \geq 0}$: **offspring distribution** or **reproduction law**.
- $m = E[X_{01}]$: **offspring mean**.
- $\sigma^2 = Var[X_{01}]$: **offspring variance**.

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## Aim of the communication

To provide robust estimators for the offspring distribution

# The problem

## Motivation

- The reproductive capacity of a small proportion of individuals can be influenced by temporary events (presence of a disease with a low prevalence, punctual changes on the environmental conditions, etc.).

**Example**: during the process of mammalian cell division, or mitosis, a mother cell divides equally into two daughter cells, but it comes to cancer, mother cells may be far more prolific.


Fig: Cell division into five daughter cells. Image credit: UCLA Engineering.

# The problem

In the frame of the branching processes, robust estimation:

- By using weighted least trimmed estimation for BGWP

STOIMENOVA, V., ATANASOV, D. AND YANEV, N. (2004) Robust estimation and simulation of branching processes. *Comptes rendus de l'Acadèmie bulgare des sciences*, **57(5)**, 19–22.

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- By considering disparity measures distance estimation in a frecuentist context for CBP

GONZÁLEZ, M, MINUESA, C. AND IP (2017) Minimum disparity estimation controlled branching process. *Electronic Journal of Statistics*, **11(1)**,295–325.

# The problem

## Assumption

The offspring distribution belongs to a parametric family

$$\mathcal{F}_\theta = \{\boldsymbol{p}_\theta : \theta \in \Theta\}, \qquad \Theta \subseteq \mathbb{R},$$

that is, $p = \boldsymbol{p}_{\theta_0}$, with $\theta_0 \in \Theta$. Moreover

$$p_k(\theta_1) = p_k(\theta_2), \quad \forall k \in \mathbb{N}_0 \qquad \Rightarrow \qquad \theta_1 = \theta_2,$$

identifiability condition.

## Aim

In a Bayesian framework, to obtain robust estimators of $\theta_0$ given the entire family tree.

- Sample: $\mathcal{Z}_n^* = \left\{ Z_l(k) = \sum_{i=1}^{\phi_l(Z_l)} I_{\{X_{li}=k\}} : k \geq 0; l = 0, \ldots, n-1 \right\}$.

# The problem

- Likelihood function of $\theta$ based on $\mathcal{Z}_n^*$:

$$f(\mathcal{Z}_n^*|\theta) = \prod_{l=0}^{n-1} \frac{\phi_l^*!}{\prod_{k=0}^{\infty} Z_l(k)} \prod_{k=0}^{\infty} p_k(\theta)^{Z_l(k)} P[\phi_l(z_l) = \phi_l^*].$$

- Posterior density:

$$\pi(\theta|\mathcal{Z}_n^*) \propto f(\mathcal{Z}_n^*|\theta)\pi(\theta) \propto \pi(\theta) \prod_{l=0}^{n-1} \prod_{k=0}^{\infty} p_k(\theta)^{Z_l(k)}.$$

- Bayesian point estimators
  - Expectation a posteriori (EAP):

$$\theta_n^* = \int_\Theta \theta\pi(\theta|\mathcal{Z}_n^*)d\theta.$$

  - Maximum a posteriori (MAP):

$$\theta_n^+ = \arg\max_{\theta\in\Theta} \pi(\theta|\mathcal{Z}_n^*).$$

# Simulated example

- **Parametric family**:

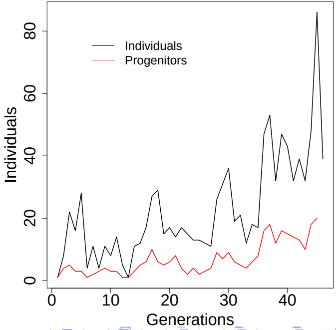
$$\mathcal{F}_\theta = \{G(\theta) : \theta \in (0,1)\}, \quad G(\theta) \equiv \text{geometric distribution with parameter } \theta.$$

- **Mixture model for gross errors**:

$$p(\theta_0, \alpha, L) = (1-\alpha)G(\theta_0) + \alpha\delta_L, \qquad \theta_0 = 0.3, \quad \alpha = 0.05, \quad L = 11.$$

We have simulated 45 generations of a CBP:

- $Z_0 = 1$ individual.
- $X_{ij} \sim p(\theta, \alpha, L)$, for $i = 0, 1, \ldots$, $j = 1, \ldots$.
- $\phi_n(k) \sim \mathcal{P}(k\lambda)$, with $\lambda = 0.6$, $k \geq 0$.
- $m = 2.333$ and $\sigma^2 = 7.778$.



# Simulated example

- **Posterior density**:

$$\pi(\theta|\mathcal{Z}_n^*) \propto \pi(\theta) \prod_{k=0}^{\infty} \prod_{l=0}^{n-1} p_k(\theta)^{Z_l(k)}.$$

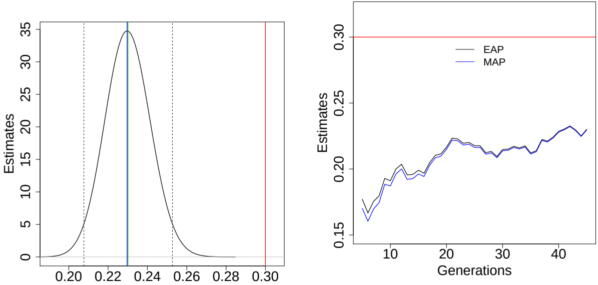



Fig: Posterior density of $\theta$ at the generation 45 (left). Temporal evolution of the EAP and MAP estimates for $\theta_0$ (right). Red lines represent the true value of the parameters and dashed lines represent the 95% HPD interval.

# Estimation via disparities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- González, M., Minuesa, C., I.P., Vidyashankar, A.N. (2017). Robust estimation in controlled branching processes: Bayesian estimators via disparities. arXiv:1802.05917.

# Bayesian robustness

It is easy to prove that

$$f(\mathcal{Z}_n^*|\theta) \propto \exp\left(\Delta_{n-1}\sum_{k=0}^{\infty}\hat{p}_{n,k}\log(p_k(\theta))\right) = \exp\left(-\Delta_{n-1}KL(\hat{p}_n,\theta)\right),$$

where

$$\begin{aligned}
\Delta_{n-1} &= \sum_{l=0}^{n-1}\phi_l(Z_l) \\
\hat{p}_{n,k} &= \frac{\sum_{l=0}^{n-1} Z_l(k)}{\Delta_{n-1}}, \quad k \geq 0, \quad (\textbf{MLE of } p \textbf{ based on } \mathcal{Z}_n^*). \\
KL(q,\theta) &= \sum_{k=0}^{\infty}\log\left(\frac{q_k}{p_k(\theta)}\right)q_k,
\end{aligned}$$

# Bayesian robustness

## Disparity measure

A **disparity measure** between $q \in \Gamma$ and $p(\theta) \in \mathcal{F}_\theta$ is defined by:

$$D(q,\theta) = \sum_{k=0}^{\infty} G(\delta(q,\theta,k))p_k(\theta),$$

with $G(\cdot)$ a three times differentiable and strictly convex function on $[-1,\infty)$ with $G(0) = 0$ and

$$\delta(q,\theta,k) = \frac{q_k}{p_k(\theta)} - 1 \quad (\textit{Pearson residual}).$$

## Examples of disparity measures

| Disparity measure | Notation | $G(\delta)$ |
|---|---|---|
| Kullback-Leibler divergence | $KL(q,\theta)$ | $(\delta+1)\log(\delta+1) - \delta$ |
| Squared Hellinger distance | $HD(q,\theta)$ | $2[(\delta+1)^{1/2} - 1]^2$ |
| Negative exponential disparity | $NED(q,\theta)$ | $\exp(-\delta) - 1 + \delta$ |

# Bayesian robustness

- Posterior density:

$$\pi(\theta|\mathcal{Z}_n^*) \propto \exp\left(-\Delta_{n-1}\mathbf{KL}(\hat{\mathbf{p}}_{\mathbf{n}},\theta)\right)\pi(\theta).$$

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- **D-Posterior density**:

$$\pi_D^n(\theta|\hat{p}_n) \propto \exp\left(-\Delta_{n-1}\mathbf{D}(\hat{\mathbf{p}}_\mathbf{n},\theta)\right)\pi(\theta).$$

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- Expectation a posteriori (EAP):

$$\theta_n^* = \int_\Theta \theta\pi(\theta|\mathcal{Z}_\mathbf{n}^*)d\theta.$$

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- **D-Posterior density**:

$$\pi_D^n(\theta|\hat{p}_n) \propto \exp\left(-\Delta_{n-1}\mathbf{D}(\hat{\mathbf{p}}_\mathbf{n},\theta)\right)\pi(\theta).$$

- **Expectation D-a posteriori (EDAP)**:

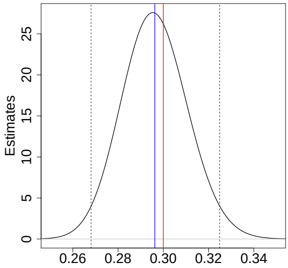
$$\theta_n^{D*} = \int_\Theta \theta\pi_\mathbf{D}^\mathbf{n}(\theta|\hat{\mathbf{p}}_\mathbf{n})d\theta.$$

- **Maximum D-a posteriori (MDAP)**:

$$\theta_n^{D+} = \arg\max_{\theta\in\Theta} \pi_\mathbf{D}^\mathbf{n}(\theta|\hat{\mathbf{p}}_\mathbf{n}).$$

# Simulated example (Continuation)

- **HD-Posterior density**: $\pi_{HD}(\theta|\hat{p}_n) \propto \pi(\theta) e^{2\Delta_{n-1}\sum_{k=0}^{\infty}(\hat{p}_{n,k}p_k(\theta))^{1/2}}$.
- **NED-Posterior density**: $\pi_{NED}(\theta|\hat{p}_n) \propto \pi(\theta) e^{-\Delta_{n-1}\sum_{k=0}^{\infty}\left(\exp\left\{-\left(\frac{\hat{p}_{n,k}}{p_k(\theta)}-1\right)\right\}-1\right)p_k(\theta)}$.



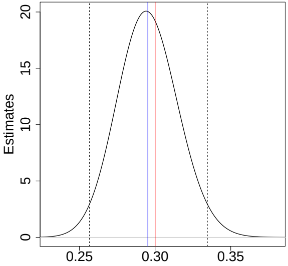




Fig: HD-Posterior density (left) and NED-Posterior density (right) of $\theta$ at the generation 45 (left). Red lines represent the true value of the parameters and dashed lines represent the 95% HPD interval.

## EDAP and MDAP functions, $\overline{T}_n$ and $\widetilde{T}_n$

For $(q, \omega) \in \Gamma \times \Omega$

$$\begin{aligned}
\overline{T}_n(q)(\omega) &= \frac{\int_\Theta \theta e^{-\Delta_{n-1}(\omega) D(q,\theta)} \pi(\theta) d\theta}{\int_\Theta e^{-\Delta_{n-1}(\omega) D(q,\theta)} \pi(\theta) d\theta} \\
\widetilde{T}_n(q)(\omega) &= \arg\min_{\theta \in \Theta} \left( \Delta_{n-1}(\omega) D(q, \theta) - \log(\pi(\theta)) \right)
\end{aligned}$$

# Bayesian robustness

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Under certain conditions it is proved that $\overline{T}_n$ and $\widetilde{T}_n$ random variables and

- $\overline{T}_n(\cdot)$ is almost surely continuous on $\widetilde{\Gamma}$ with respect to the $l_1$-metric; that is, $q_j \to q$ in $l_1$, then $\overline{T}_n(q_j) \to \overline{T}_n(q)$, as $j \to \infty$, with probability one.

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- The function $\widetilde{T}_n(\cdot)$ is continuous in $q$; that is, $\widetilde{T}_n(q_j) \to \widetilde{T}_n(q)$ with probability one as $j \to \infty$, as $q_j \to q$ in the sense that $\sup_{\theta \in \Theta} |D(q_j, \theta) - D(q, \theta)| \to 0$.

## Relationship EDAP and MDAP functions with their frequentist counterpart

The minimum disparity estimator (MDE) of $\theta_0$ based on $\hat{p}_n$, which is defined as:

$$\hat{\theta}_n^D = \arg\min_{\theta \in \Theta} D(\hat{p}_n, \theta),$$

and the associated *disparity function* defined as:

$$\begin{aligned} T: \quad \Gamma &\rightarrow \Theta \\ q &\mapsto T(q) = \arg\min_{\theta \in \Theta} D(q, \theta), \end{aligned}$$

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Under certain conditions, it can be proved, on $\{Z_n \rightarrow \infty\}$:

- $\overline{T}_n(q) - T(q) = o\left(\Delta_{n-1}^{-1/2}\right)$ a.s.
- $\widetilde{T}_n(q) - T(q) = o\left(\Delta_{n-1}^{-1/2}\right)$ a.s.

# Bayesian robustness estimators: asymptotic properties

## Some notation

- $I^D(\theta) = \ddot{D}(p, \theta)$, and $I_n^D(\theta) = \ddot{D}(\hat{p}_n, \theta)$, where recall that $p$ is the posited offspring distribution and $\hat{p}_n$ is the MLE
- Thus, $p = \boldsymbol{p}_{\theta_0}$, one has that $I^D(\theta_0)$ reduces to the Fisher information at $\theta_0$ denoted by $I(\theta_0)$
- $\varphi(t;\theta)$ denotes the density function of a normal distribution with mean 0 and variance $I^D(\theta)^{-1}$
- $\varphi_n(t)$ denotes the density function of a normal distribution with mean 0 and variance $I_n^D(\hat{\theta}_n^D)^{-1}$.

# Bayesian robustness estimators: asymptotic properties

Let $\overline{\pi}_D^n(\cdot|\hat{p}_n)$ denote the $D$-posterior density function of $t = \Delta_{n-1}^{1/2}(\theta - \hat{\theta}_n^D)$. Under some regularity conditions, on $\{Z_n \to \infty\}$, then:

1. $\int |\overline{\pi}_D^n(t|\hat{p}_n) - \varphi(t;\theta_p)|dt \to 0 \quad a.s.$
2. $\int |t||\overline{\pi}_D^n(t|\hat{p}_n) - \varphi(t;\theta_p)|dt \to 0 \quad a.s.$
3. $\int |\overline{\pi}_D^n(t|\hat{p}_n) - \varphi_n(t)|dt \to 0 \quad a.s.$
4. $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} |\overline{\pi}_D^n(t|\hat{p}_n) - \varphi(t;\theta_p)| = 0 \quad$ a.s.

# Bayesian robustness estimators: asymptotic properties

- **Strong consistency of EDAP**:

$$\theta_n^{D*} \xrightarrow[n\to\infty]{a.s.} \theta_0, \quad \text{on } \{Z_n \to \infty\}.$$

- **Asymptotic normality of EDAP**:

$$\Delta_{n-1}^{1/2}(\theta_n^{D*} - \theta_0) \xrightarrow[n\to\infty]{d} N(0, I(\theta_0)^{-1}), \quad \text{on } \{Z_n \to \infty\}.$$

- **Strong consistency of MDAP**:

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# Simulated example (Continuation)

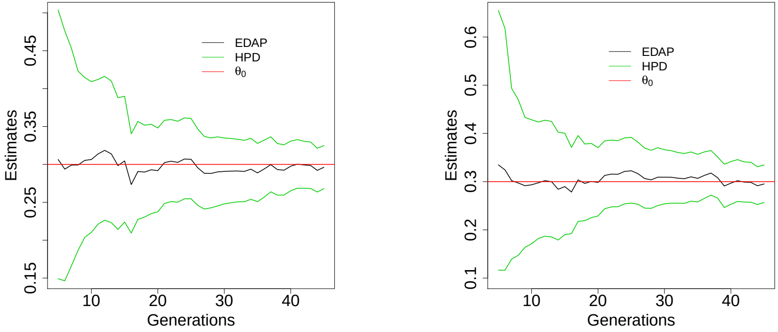


Fig: EDAP estimates (black lines) for the HD (left) and NED (right), with the 95% HPD intervals (green lines) and true value of $\theta$ (red lines).

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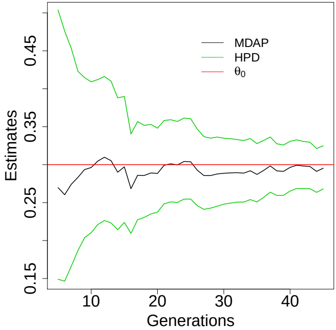


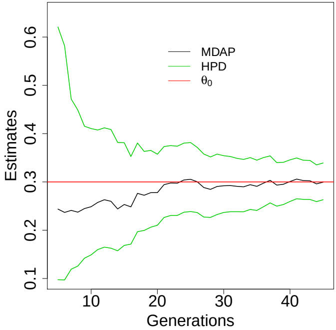


Fig: MDAP estimates (black lines) for the HD (left) and NED (right), with the 95% HPD intervals (green lines) and true value of $\theta$ (red lines).

# Robust properties

✓ We focus on the gross error contamination model given by

$$p(\theta, \alpha, L) = (1-\alpha)\boldsymbol{p}_\theta + \alpha\eta_L, \tag{1}$$

where $\theta \in \Theta$, $\alpha \in (0,1)$, $L \in \mathbb{N}_0$, and $\eta_L$ is a point mass distribution at $L$.

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✓ We define $\alpha$-influence function of a random variable $\overline{T} : \Gamma \times \Omega \to \Theta$. For $\alpha \in (0,1)$, set

$$\begin{aligned} IF_\alpha(\cdot, \overline{T}, p) : \mathbb{N}_0 \times \Omega &\to \mathbb{R} \\ (L, \omega) &\mapsto IF_\alpha(L, \overline{T}, p)(\omega) = \frac{\overline{T}(p(\theta_0, \alpha, L))(\omega) - \overline{T}(\boldsymbol{p}_{\theta_0})(\omega)}{\alpha}. \end{aligned}$$

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✓ The influence function for EDAP estimators at $p$ is given by

$$\begin{aligned} IF(\cdot, \overline{T}_n, p) : \mathbb{N}_0 &\to \mathbb{R} \\ L &\mapsto IF(L, \overline{T}_n, p) = \lim_{\alpha \to 0} IF_\alpha(L, \overline{T}_n, p). \end{aligned}$$

Under some conditions $|IF(L, \overline{T}_n, p)| < \infty$, for each $L \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

Study of breakdown point

✓ Classically, the breakdown point of a general function $\overline{T}$ at $q \in \Gamma$ is defined as:

$$B(\overline{T}, q) = \sup\{\alpha \in (0,1) : b(\alpha, \overline{T}, q) < \infty\},$$

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✓ Under some regularity conditions, the breakdown points of the EDAP and MDAP functions at $p$ are 1, respectively.

# Concluding remarks

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- We have **implemented** this methodology using statistical software and programming environment **R**.

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