Asymptotic properties of expansive Galton-Watson trees

JEAN-FRANÇOIS DELMAS

http://cermics.enpc.fr/~delmas

CERMICS, École des Ponts, Univ. Paris-Est

Branching processes and their applications
Badajoz 2018

Outline

In collaboration with R. Abraham (ArXiv 2017) and with R. Abraham and A. Bouaziz (ArXiv 2017).
We shall forget about any peridioc condition in this presentation.

Local limit

- $\mathbb{T}$ set of rooted ordered trees.
- For $\mathbf{t} \in \mathbb{T}$, set $|\mathbf{t}|$ its cardinal, $H(\mathbf{t})$ its height.
- For $v \in \mathbf{t}$, set $k_v(\mathbf{t}) \in \mathbb{N} \bigcup \{\infty\}$ its out-degree and $H(v)$ its height.
- Subset of trees with finite out-degrees and finite height $h \in \mathbb{N}$:

$$\mathbb{T}_{\mathrm{f}} = \{\mathbf{t} \in \mathbb{T},\ k_v(\mathbf{t}) < \infty \text{ for all } v \in \mathbf{t}\} \quad \text{and} \quad \mathbb{T}_{\mathrm{f}}^{(h)} = \{\mathbf{t} \in \mathbb{T}_{\mathrm{f}},\ H(\mathbf{t}) = h\}.$$

- Truncation at level $h \in \mathbb{N}^*$: $r_h(\mathbf{t}) = \{v \in \mathbf{t},\ H(v) \leq h\}$.
- A sequence $(T_n, n \in \mathbb{N})$ of random trees converges locally in law towards a random tree $T \in \mathbb{T}_{\mathrm{f}}$ (write $T_n \xrightarrow[n\to\infty]{\textbf{(d)}} T$) iff:

$$\lim_{n\to\infty} \mathbb{P}(r_h(T_n) = \mathbf{t}) = \mathbb{P}(r_h(T) = \mathbf{t}) \quad \text{for all } h \in \mathbb{N}^*,\ \mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}.$$

- Convergence in the condensation case (that is $T \notin \mathbb{T}_{\mathrm{f}}$) is more technical.

GW trees

- Offspring distribution: $p = (p(k),\, k \in \mathbb{N})$.
- Assume p is non degenerate: $\#\{k \in \mathbb{N},\, p(k) > 0\} \geq 2$.
- Assume finite mean: $\mu = \sum_{k\in\mathbb{N}} kp(k) \in (0, +\infty)$.
- GW tree τ is a random tree with distribution defined by:

$$\mathbb{P}(r_h(\tau) = \mathbf{t}) = \prod_{v\in\mathbf{t},\, H(v)<h} p(k_v(\mathbf{t})) \quad \text{for all } h \in \mathbb{N}^*,\, \mathbf{t} \in \mathbb{T}_\mathrm{f}^{(h)}.$$

GW process

- Size of the population at height h: $z_h(\mathbf{t}) = \#\{v \in \mathbf{t},\, H(v) = h\}$.
  Process $z(\mathbf{t}) = (z_h(\mathbf{t}), h \in \mathbb{N})$.
- $z(\tau)$ is the GW process associated to the GW tree τ.
- Let $\tau_n \overset{\text{(d)}}{=} \tau$ cond. on some event $\mathcal{E}_n$ s.t. $\mathbb{P}(\mathcal{E}_n) > 0$, for $n \in \mathbb{N}^*$.
- Well known example: $\mathcal{E}_n = \{z_n(\tau) > 0\}$.

- Aim: existence and representation of the local limit of $(\tau_n, n \in \mathbb{N}^*)$.

- Why consider GW trees τ_n instead of GW processes $z(\tau_n)$?

Extinction probability $\mathfrak{c}$ and representation of super-critical GW tree

- Set $\mathfrak{c}$ the extinction probability: $\mathfrak{c} = \mathbb{P}(H(\tau) < \infty)$.
- If $\mathfrak{c} > 0$, set $p' = (p'(k) = \mathfrak{c}^{k-1} p(k), k \in \mathbb{N})$.
- If $\mu \leq 1$ and $\mathfrak{c} > 0$, then $\mathfrak{c} = 1$ and $p' = p$!

- If $\mu > 1$ and $\mathfrak{c} > 0$, then $\mathfrak{c} \in (0, 1)$ and τ has a two-type representation:
  - The vertices are either of type s (for survivor) or of type e (for extinction).
  - Root is of type s with probability $1 - \mathfrak{c}$.
  - Branching property holds for type e and s.
  - A vertex of type e produces vertices of type e with offspring distrib. p'.
  - A vertex of type s produces vertices of type s (at least one) and of type e.

Kesten tree τ^0

- If $\mathfrak{c} > 0$: Kesten tree τ^0 seen as a two-type GW tree:
  - The vertices are either of type s (for survivor) or of type e (for extinction).
  - Root is of type s.
  - Branching property holds for type e and s.
  - A vertex of type e produces vertices of type e with offspring distrib. p'.
  - A vertex of type s produces one vertex of type s and vertices of type e.

- If $\mathfrak{c} = 0$, the (degenerate) Kesten tree τ^0 is the regular $\mathfrak{a}$-ary tree, with:

$$\mathfrak{a} = \inf\{k,\, p(k) > 0\} \geq 1.$$

Local convergence to Kesten tree for (sub-)critical GW trees

Recall: $\tau_n \overset{(d)}{=} \tau$ conditionally on some event $\mathcal{E}_n$, s.t. $\mathbb{P}(\mathcal{E}_n) > 0$.

- **Case** $\mu = 1$. Taking $\mathcal{E}_n$ equal to:
  - $\{H(\tau) = n\}$;
  - $\{|\tau| = n\}$ or more generally $\{\#\{v \in \tau,\, k_v(\tau) \in A\} = n\}$ with $A \subset \mathbb{N}$;

  we get:

$$\tau_n \xrightarrow[n\to\infty]{(d)} \tau^0. \tag{1}$$

- **Case** $\mu < 1$. For $\mathcal{E}_n = \{H(\tau) = n\}$, we get also (1).
- **Case** $\mu < 1$. For $\mathcal{E}_n = \{|\tau| = n\}$, the limit is:
  - either a **Kesten tree** (but associated to an offspring distribution $\neq p$)
  - or a random tree with **one vertex with infinite out-degree** (condensation).

- Can we get other limits (such as infinite backbone?)

The geometric case (Abraham-Bouaziz-D. (2017))

Consider:

- $\mathcal{E}_n = \{z_n(\tau) = a_n\}$ with $a_n \geq 1$.
- Geometric case: $p(0) = 1 - \eta$ and $p(k) = \eta q(1-q)^{k-1}$, $k \in \mathbb{N}^*$.
- Assume $\mu = \eta/q \in (1, +\infty)$.

For $\lim_{n\to\infty} a_n/\mu^n = \theta \in [0, +\infty]$, we have:

$$\tau_n \xrightarrow[n\to\infty]{\text{(d)}} \tau^\theta, \tag{2}$$

- for $\theta = 0$, τ^0 is the Kesten tree (infinite spine);
- for $\theta \in (0, +\infty)$, τ^θ has an infinite backbone;
- for $\theta = +\infty$, τ^∞ exhibits condensation at the root (only).

Furthermore $(\tau^\theta, \theta \in [0, \infty])$ is continuous in distribution.
Also (2) holds for $\mu < 1$ with $c_n = \mu^{-n}$, and for $\mu = 1$ with $c_n = n^2$.

Representation of τ^θ

Assume $\theta \in (0, +\infty)$. τ^θ seen as a two-type GW tree.

- The vertices are of type s (for survivor) or of type e (for extinction).
- Root is of type s.
- Branching property holds for vertices of type e ONLY.
- A vertex of type e produces vertices of type e with offspring distrib. p'.
- If $\mu = 1$: at generation h a parent of type s has a child of type s and there is a Poisson immigration with parameter $\theta q/(1-q)$ of individuals of type s which are grafted uniformly on all parents of type s. Then add vertices of type e to parents of type s.
- If $\mu \neq 1$: same spirit but with an immigration rate at generation h which is increasing with h.

General super-critical case (Abraham-D. (2017))

Assume $\mu \in (1, \infty)$. Set $\mathcal{E}_n = \{z_n(\tau) = a_n\}$ with $a_n \geq 1$.
Recall there exists a sequence $(c_n, n \in \mathbb{N})$ such that:

$$z_n(\tau)/c_n \xrightarrow[n\to\infty]{\text{a.s.}} W,$$

with $\mathbb{P}(W = 0) = \mathfrak{c}$. (Under the $L\log(L)$ cond., take $c_n = \mu^n$.)
For $\lim_{n\to\infty} a_n/c_n = \theta \in [0, +\infty)$, we have:

$$\tau_n \xrightarrow[n\to\infty]{\text{(d)}} \tau^\theta, \tag{3}$$

- for $\theta = 0$, τ^0 is the Kesten tree (infinite spine or $\mathfrak{a}$-ary tree);
- for $\theta \in (0, +\infty)$, τ^θ has an infinite backbone.
- For $\theta = +\infty$, (3) holds if $\mathfrak{b} = \sup\{k,\ p(k) > 0\} < \infty$ and τ^∞ is then the $\mathfrak{b}$-ary tree.

Furthermore $(\tau^\theta, \theta \in [0, \infty))$ is

- continuous in distribution (for the local convergence) ;
- the regular distribution of τ conditionally on $\{W = \theta\}$.

Idea of the proof

For $h \in \mathbb{N}$ and $\mathbf{t} \in \mathbb{T}_{\mathrm{f}}^{(h)}$, with $k = z_h(\mathbf{t})$:

$$\mathbb{P}(r_h(\tau_n) = \mathbf{t}) = H_n(h,k)\mathbb{P}(r_h(\tau) = \mathbf{t}) \quad \text{with} \quad H_n(h,k) = \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}_1(Z_n = a_n)}$$

and $Z = (Z_n, n \in \mathbb{N})$ is under $\mathbb{P}_k$ a GW process started from $Z_0 = k$.

- The local cv of τ_n towards τ^θ is equivalent to the cv of H_n to some H^θ.
- Precise asymptotics of $\mathbb{P}_k(Z_{n-h} = a_n)$ for $\theta \in (0, +\infty)$ are given by Dubuc and Seneta (1976).
- Precise asymptotics of $\mathbb{P}_k(Z_{n-h} = a_n)$ for $\theta = 0$ are given by Fleischmann and Wachtel (2007) for the Schröder case $\mathfrak{a} \leq 1$; and (2009) for the Böttcher case $\mathfrak{a} \geq 2$.
- Precise asymptotics of $\mathbb{P}_k(Z_{n-h} = a_n)$ for $\theta = +\infty$ in the Harris case $\mathfrak{b} < \infty$ in the same spirit as for the Böttcher case.

Extremal martingale and Martin boundary (super-crit. case)

- Aim: describe $\mathcal{M}$ the family of extremal non-negative space-time harmonic functions (Martin boundary for the space-time GW process).
- For $n \geq h, k \in \mathbb{N}$, we set for $\lim_{n\to\infty} a_n/c_n = \theta \in [0, +\infty]$:

$$H_n(h,k) = \frac{\mathbb{P}_k(Z_{n-h} = a_n)}{\mathbb{P}_1(Z_n = a_n)} \quad \text{and, if it exists,} \quad H^\theta(h,k) = \lim_{n\to\infty} H_n(h,k).$$

- If $\mathfrak{c} > 0$, set $H^{0,0} = \lim_{n\to\infty} H_n$, with $a_n = 0$, which is well defined.
- Dynkin (1969): $\mathcal{M}$ is a subset of all the possible limits of H_n (for all choices of sequences (a_n) s.t. H_n converges).
- $\{H^\theta,\ \theta \in (0,\infty)\} \subset \mathcal{M}$, see Athreya and Ney (1970) and Kemeny, Snell and Knapp (1976).

Conclusion

g generating function of p and w_k the density of W when $Z_0 = k$.

$$H^{0,0}(h,k) = \mathfrak{c}^{k-1}.$$

$$H^{0}(h,k) = \begin{cases} k\mathfrak{c}^{k-1} g'(\mathfrak{c})^{-h} & \text{if } \mathfrak{a} = 0, \\ g'(\mathfrak{c})^{-h} \mathbf{1}_{\{k=1\}} & \text{if } \mathfrak{a} = 1, \\ p(\mathfrak{a})^{-(\mathfrak{a}^h - 1)/(a-1)} \mathbf{1}_{\{k=\mathfrak{a}^h\}} & \text{if } \mathfrak{a} \geq 2. \end{cases}$$

$$H^{\theta}(h,k) = \mu^h \frac{w_k(\mu^h \theta)}{w_1(\theta)} \qquad \text{for } \theta \in (0, +\infty).$$

$$H^{\infty}(h,k) = \begin{cases} p(\mathfrak{b})^{-(\mathfrak{b}^h - 1)/(\mathfrak{b}-1)} \mathbf{1}_{\{k=\mathfrak{b}^h\}} & \text{if } \mathfrak{b} < \infty, \\ 0 & \text{if } p \text{ is geometric.} \end{cases}$$

And, if $\mathfrak{b} < \infty$ or p geometric, we have:

$$\mathcal{M} = \{H^{0,0}\} \bigcup \{H^{\theta},\ \theta \in [0, +\infty]\} \quad \text{(up to the 0 function).}$$

Conjectures and open questions

g generating function of p and $R = \inf\{r \geq 1,\ g(r) < +\infty\}$.

If $\mu > 1$ and $\theta = +\infty$:

- If $R = +\infty$: we conjecture that τ^∞ has no condensation and H^∞ is non-zero. (Similar to the Harris case $\mathfrak{b} < \infty$.)
- If $R < +\infty$ and $g(R-) = +\infty$: we conjecture that the root of τ^∞ has infinite out-degree a.s. and $H^\infty = 0$. (Similar to the geometrical case.)
- If $R < +\infty$ and $g(R-) < +\infty$: open question.

If $\mu \leq 1$:

- If $\mu < 1$, when possible see τ as a super-critical GW tree conditioned to extinction, and then transfer the results.
  If this is not possible, then open question for all sequences (a_n).
- $\mu = 1$: open question for all sequences (a_n).