Immigration and Cohabitation in Resource-Dependent Branching Processes

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Assumptions throughout

All random variables are supposed to have finite 2nd moments

independence "within" sub-populations

RDBP-setting with immigration

Home-population $\longleftarrow$ **immigrants**

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Create a mathematical framework to study the question:

How to enable an equilibrium between them?

Def.: Let $(\Gamma_t) = (\Gamma_t^h, \Gamma_t^i)_{t=1,2,,\cdots}$ be a bi-variate stochastic counting process. We say (Γ_t) converges to an *equilibrium* if

$$\exists\, 0 < \alpha < \infty : P\left(\frac{\Gamma_t^i}{\Gamma_t^h} \;\rightarrow\; \alpha \,\middle|\, \Gamma_t^i \not\rightarrow 0,\; \Gamma_t^h \not\rightarrow 0\right) = 1.$$

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Problem: Find conditions (necessary, sufficient) for existence of equilibrium.

B-Robertson-Steele-equation and BRS-inequality

Let $X_1, X_2, \cdots X_n$ with continuous distr. $F_k, k = 1, \cdots, n$ with order statistics $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$.

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$$N(n,s) := \begin{cases} 0, \text{ if } X_{1,n} > s, \\ \sup\{1 \le k \le n : X_{1,n} + X_{2,n} + \cdots + X_{k,n} \le s\}, \text{ otherwise.} \end{cases}$$

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Theorem (Bruss and Robertson (1991), J. M. Steele (2016))

$$\text{(i)} \quad \mathbb{E}(N(n,s)) \le \sum_{k=1}^{n} F_k(\tau),$$

where $\tau := \tau(n,s)$ solves

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(ii) If, moreover, the X_k's are i.i.d. $F_k = F$, then

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Theorem 1: If all "macro-parameters" stay invariant over all generations, then an α-equilibrium can only exist if

$$m_h F_h(\tau) = m_i F_i(\tau) \geq 1, \tag{1}$$

where τ is the unique solution of

$$m_h \int_0^\tau x dF_h(x) + \alpha\, m_i \int_0^\tau x dF_i(x) = r_h + \alpha r_i. \tag{2}$$

Sufficiency paradox?

Solve for α

$$\alpha = \alpha(\tau) = \frac{r_h - m_h \int_0^\tau x dF_h(x)}{m_i \int_0^\tau x dF_i(x) - r_i}. \tag{3}$$

We conclude: An equilibrium can only exist if ....

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- $Y_t^{(1)} = \frac{Z_t^{(1)}}{m_1^t},\ \ Y_t^{(2)} = \frac{Z_t^{(2)}}{m_2^t}$ a.s.-converging martingales

Random equilibrium

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- If $m_1 = m_2$ then

$$(Y_t^{(1)}/Y_t^{(2)}) = (Z_t^{(1)}/Z_t^{(2)})$$

Once $(Z_t^{(1)})$ and $(Z_t^{(2)})$ sufficiently large ....SLLN ......

$$\frac{Z_{t+k}^{(1)}}{Z_{t+k}^{(2)}} \sim \frac{Z_t^{(1)} m_1^k}{Z_t^{(2)} m_1^k} = \frac{Z_t^{(1)}}{Z_t^{(2)}}, \quad \text{as } k \to \infty \tag{4}$$

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• Hence $Z_{t+k}^{(1)}/Z_{t+k}^{(2)},\ k = 1, 2, \ldots$ given $Z_t^{(1)}$ and Z_t^2 concentrates around $Z_t^{(1)}/Z_t^{(2)}$, as $k \to \infty$.

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• Early history of states of the two Galton-Watson processes points quickly to the relevant neighbourhood of the equilibrium $\alpha_Y = \lim_{t\to\infty} Y_t^{(1)}/Y_t^{(2)}$

4. "fractional" integration φ per generation

Theorem 2 For an equilibrium of the φ-integrated process to exist it is necessary that there exists values $\tau > 0$ and α_φ with $0 < \alpha_\varphi < \infty$ satisfying the equation

$$m_h\Big(1 + \varphi\alpha_\varphi\Big)\int_0^\tau x dF_h(x) + m_i\alpha_\varphi\Big(1 - \varphi\Big)\int_0^\tau x dF_i(x)$$

$$= r_h + r_i\alpha_\varphi + \varphi\alpha_\varphi\Big(r_h - r_i\Big) \qquad (5)$$

subject to the constraints

$$m_h(1 + \alpha_\varphi\varphi)F_h(\tau) = m_i(1 - \alpha_\varphi\varphi)F_i(\tau) \geq 1. \qquad (6)$$

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Resource Dependent Branching Process (RDBP)

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Bruss and Duerinckx (2015) *RDBPs and ....* ,Annals of Appl. Probab., Vol. 25, Nr 1, 324-372.

Bruss (2016) *The Theorem of Envelopment ...*, in Springer Lecture Notes in Statistics, (I.M. del Puerto et al., Eds), Vol. 219, 119-136.

Ingredients of proofs and connections

- Extinction criteria for modified GWPs (Sevast'yanov, Zubkov), Φ-BP's (Yanev)

- Borel-Cantelli type arguments, complete convergence, a.s.-convergence, average reproduction mean,

- Theorem of envelopment for RDBPs

- Bruss-Robertson-Steele/equation/inequality

Connections:

- Behaviour of populations near criticality (Afana'sev ...Vatutin, Jagers, Klebaner)

- Multi-type BPs, critical case (Dyakanova, Vatutin)

- Random Environment BPs, large deviations (Quansheng Liu)

- Controlled BPs near criticality (Inés del Puerto)

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- prefer (usually) an increasing standard of living.

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5.3 "Functioning" of society structure

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If yes, then ...., else ....

5.5 Policies Society form

$L(t) := \{X_1, X_2, \cdots, X_{D(t)}\}$ list of claims at time t

$R(t) :=$ total resource space available at time t

Weakest-first society: Selects sequentially increasing order statistics $X_{<1,D(t)>}, X_{<2,D(t)>}, \cdots$ as long as sum $\leq R(t)$

Strongest-first society:Selects sequentially decreasing order statistics $X_{<D(t),D(t)>}, X_{<D(t)-1,D(t)>}, \cdots$ as long as sum $\leq R(t)$

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optimal control for policies?