Scaling limits for general finite dimensional population models

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10th april, Badajoz

Fixed environment without interactions

Let Z be a Galton Watson process :

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_{i,n}$$

where $(L_{i,n} : i, n \geq 0)$ are i.i.d. and distributed as $L \in \mathbb{N}$.

Let $N \in \mathbb{N}$ and consider the Galton Watson process Z^N with reproduction random variable L^N and $Z_0^N = [z_0 N]$ and

$$Y_t^N = \frac{1}{N} Z_{[v_N t]}^N,$$

for $t \geq 0$, where $v_N \to \infty$ gives the time scale.

When does Y^N converge in $\mathbb{D}(\mathbb{R}^+, [0, \infty])$? To which object ?

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Solved by Helland, Lamperti (time change of a random walk) and Grimwall (tightness + convergence of generating function).

Necessary and sufficient condition :

- convergence of the random walk with step L^N to a Lévy process
- convergence of triangular arrays
- $$\lim_{N\to\infty} v_N N\,\mathbb{E}\left(h_d((L^N-1)/N)\right) = \alpha_d;$$
  $$\lim_{N\to\infty} v_N N\,\mathbb{E}\left(h_d^2((L^N-1)/N)\right) = \beta_d;$$
  $$\lim_{N\to\infty} v_N N\,\mathbb{E}\left(f((L^N-1)/N)\right) = \int_0^\infty f\nu_d,$$

  for f continuous bounded and null in a neighborhood of 0, where h_d is a truncation function and $\int_{(0,\infty)}(1\wedge v^2)\,\nu_d(dv) < \infty$.

The limiting object is a *Continuous State Branching Processes* (CSBP)

- Time change of a Lévy process (Lamperti transform)
- Unique pathwise solution of the following SDE (Fu & Li)

$$\begin{aligned} Y_t &= z_0 + \alpha_d \int_0^t Y_s ds + \sigma_d \int_0^t \sqrt{Y_s} dB_s^d + \\ &+ \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \leq Y_{s-}} h_d(z) \widetilde{N}^d + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \leq Y_{s-}} (z - h_d(z)) N^d, \end{aligned}$$

where B is a brownian motion, N^d is a Poisson measure on $(\mathbb{R}^+)^3$ with intensity $dtdz\nu_d(d\theta)$ and $\sigma_d^2 = \beta_d - \int_{(0,\infty)} h_d^2 \nu_d$.

Branching processes in random environment

We consider a sequence of random environments $(\mathcal{E}_k^N : k \geq 0)$ and

$$Z_{n+1}^N = \sum_{i=1}^{Z_n^N} L_{i,n}^N(\mathcal{E}_n^N)$$

where for each environment e, $(L_{i,n}^N(e) : i \geq 1, n \geq 0)$ are i.i.d. and distributed as a random variable $L(e) \in \mathbb{N}$ a.s.

What about the scaling limits of

$$Y_t^N = \frac{1}{N} Z_{[v_N t]}^N ??$$

Convergence

$$Z_{n+1}^N = \sum_{i=1}^{Z_n^N} L_{i,n}^N(\mathcal{E}_n^N)$$

Weak convergence of $Y^N = Z_{[v_N.]}^N/N$ to a *CSBP in random environment*.

- In some stable cases (stable branching mechanism), time change techniques of Kurtz (generalized by Borovkov)
- with finite variations of the limiting drift $t \to \alpha_t$:

$$Y_t = z_0 + \int_0^t Y_s d\alpha_s + \int_0^t \sqrt{Y_s}\sigma_s dB_s^d + \text{demographical jumps}$$

characterization of the dual problem (quenched Laplace exponent using quenched branching property) by Bansaye and Simatos.

Let us take into account density dependance for reproduction laws

$$Z_{n+1}^N = \sum_{i=1}^{Z_n^N} L_{i,n}^N(Z_n^N)$$

where for each size z, $(L_{i,n}^N(z) : i \geq 1, n \geq 0)$ are i.i.d. and distributed as a random variable $L(z) \in \mathbb{N}$ a.s.

Weak convergence to *CSBP with interactions* of the form

$$\begin{aligned} Y_t &= z_0 + \int_0^t {\color{blue}g}(Y_s)ds + \sigma_d \int_0^t \sqrt{Y_s} dB_s^d + \\ &+ \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \leq Y_{s-}} h_d(z)\widetilde{N}^d + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \leq Y_{s-}} (z - h_d(z))N^d, \end{aligned}$$

see works of Pardoux & Dramé for some class in continuous time *with moment assumptions* (tightness + martingale problem).

Problem : the branching property fails and without stability no relevant time change found.

Objectives :

- go beyond these assumptions (finite variations, moment assumptions, stability...) and extend the general criterion of convergence of Galton-Watson processes
- capture more complex population structures (with several species : competition, predation, sexe, mutations...)

for processes of the form

$$\begin{cases} Z_{n+1}^N = \sum_{i=1}^{F_N(Z_n^N)} L_{i,n}^N(Z_n^N, E_n^N), \\ \\ S_{n+1}^N = S_n^N + E_n^N \end{cases}$$

where for each (z, e), $(L_{i,n}^N(z, e) : i \geq 1, n \geq 0)$ are i.i.d. and distributed as a random variable $L(z, e) \in \mathbb{N}$ a.s. and $(E_n^N : n \geq 0)$ are i.i.d.

The Markov chain (Z^N, S^N) is characterized by its transition and roughly its law is given by the conditional increments :

$$\mathbb{E}(H(Z_1^N - z, E_0^N)|Z_0^N = z)$$

for a rich enough class of functions H.

For convergence of scaled Markov chains, this can be made rigorous by considering the convergence of the *characteristics of the semi martingale*

$$(Z_{[v_N.]}^N/N, S_{[v_N.]}^N).$$

[Jacod Shirayev] : one then needs to focus on a class of functions H null in zero containing a truncation function, its square and functions null in a neighborhood of zero.

$$v_N\mathbb{E}(H(Z_1^N/N - z, E_0^N)|Z_0^N = Nz)$$

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Convergence of the linear operator $H \to \mathcal{G}^N H$:

$$\mathcal{G}^N H(z) = v_N \mathbb{E}\left(H(Z_1^N/N - z, E_0^N) | Z_0^N = Nz\right)$$

To exploit the (conditional !) independence structure in the individual based model, one may prefer to focus on functions of the form

$$H_{k,\ell}(u, v) = 1 - e^{-ku-\ell v}$$

(rich since its generates an algebra separating points).
We obtain

$$\mathcal{G}^N_{k,\ell} H(z) = v_N \left(1 - \mathbb{E}\left(e^{-\ell E_0^N} P_k^N(z, E_0^N)^{F_N(Nz)}\right)\right)$$

with

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captured via the convergence of functions

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as $N \to \infty$ with

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- ☺ generalizes the necessary and sufficient condition for Galton-Watson processes
- ☹ uniformity with respect to z is required in the convergence to apply the technics (Taylor expansions and some analysis involved using convexity or boundedness argument).

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Reduction to $\mathcal{H}$

Let X^N be a sequence Markov chains taking values in a subset of $\mathbb{R}^d$. Let $\mathcal{H}$ be a functional space which is dense in the set of regular functions null at zero for a norm equivalent to $\|H\| = \| u \to H(u)/(1 \wedge u^2) \|_\infty$ and

$$\mathcal{G}_x^N(H) = v_N \mathbb{E}\big(H(X_1^N - X_0^N) \,|\, X_0^N = x\big),$$

- If $\mathcal{G}_.^N$ is bounded and converges uniformly to $\mathcal{G}_. H$ for any $H \in \mathcal{H}$, then $X_{[v_N .]}^N$ is tight.
- If the limiting operator $\mathcal{G}_. H$ is continuous for any $H \in \mathcal{H}$, then any limiting value of X^N is a semimartingale whose characteristics are determined by $(\mathcal{G}_. H : H \in \mathcal{H})$
- If uniqueness holds for the associated SDE, then X^N converges to the solution of the SDE.

Let

$$\begin{cases} Z_{n+1}^N = \sum_{i=1}^{Z_n^N} L_{i,n}^N(Z_n^N, E_n^N), \\ S_{n+1}^N = S_n^N + E_n^N \end{cases}$$

Consider

$$X_n^N = \left(e^{-Z_n^N/N}, S_n^N\right)$$

and

$$\mathcal{H} = \{(v, w) \to v^k e^{-\ell w} : k \geq 1, \ell \geq 0\} \cup \{(v, w) \to 1 - e^{-\ell w} : \ell \geq 1\}.$$

(local Stone Weierstrass theorem for the density) to get tightness and identification.

Here pathwise uniqueness of the SDE obtained applying Pu & Li (see also Palau & Pardo Millan).

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Examples

- Convergence of cooperative GW processes to *cooperative CSBP* (potentially explosive) :

$$
\begin{aligned}
Z_t &= z_0 + \alpha_d \int_0^t Z_s ds + \int_0^t Z_s g(Z_s) ds + \sigma_d \int_0^t \sqrt{Z_s} dB_s^d \\
&+ \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le Z_{s-}} h_d(z) \widetilde{N}^d + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le Z_{s-}} (z - h_d(z)) N^d
\end{aligned}
$$

  when g is regular and does not tend too fast to infinity.

- Convergence of Galton Watson process in random environment with competition to *logistic Feller diffusion in a Brownian environment*

$$
Z_t = z_0 + \alpha_d \int_0^t Z_s ds - c \int_0^t Z_s^2 ds + \sigma_e \int_0^t Z_s dB_s^e + \sigma_d \int_0^t \sqrt{Z_s} dB_s^d .
$$

Wright Fisher in a Lévy environment

For each $N \geq 1$, it is recursively defined for $n \geq 0$ by

$$\begin{cases} Z_{n+1}^N = \sum_{i=1}^N B_{n,i}^N(Z_n^N/N, \mathcal{E}_n^N), \\ \\ S_{k+1}^N = S_k^N + \mathcal{E}_k^N, \end{cases}$$

and $(\mathcal{E}_k^N)_k$ are i.i.d. with values in $(-1, +\infty)$ and $\left(B_{k,i}^N(z,e); k \geq 1, i \geq 1\right)$ are Bernoulli random variable $\mathcal{E}^N(z,e)$ defined by

$$\mathbb{P}(B^N(z,e) = 1) = p(z,e) \; ; \; \mathbb{P}(B^N(z,e) = 0) = 1 - p(z,e).$$

In particular, Wright Fisher diffusion with selection in a Lévy environment

$$p(z,e) = \frac{z(1+e)}{z(1+e) + 1 - z}$$

Multi dimensional model (bisexual Galton Watson)

Consider a bisexual Galton-Watson model with aging and classical monogamous mating (with mutual fidelity)

$$\begin{cases} F_{n+1}^N = \sum_{i=1}^{F_n^N} \mathcal{E}_{n,i}^{f,N} + \sum_{j=1}^{M_n^N \wedge F_n^N} L_{n,j}^{f,N}, \\ \\ M_{n+1}^N = \sum_{i=1}^{M_n^N} \mathcal{E}_{n,i}^{m,N} + \sum_{j=1}^{M_n^N \wedge F_n^N} L_{n,j}^{m,N}, \end{cases}$$

Assume for $\bullet \in \{f, m\}$,

$$\lim_{N\to\infty} v_N N \,\mathbb{E}(h(L_\bullet^N/N)) = \alpha_\bullet; \qquad \lim_{N\to\infty} v_N N \,\mathbb{E}(h^2(L_\bullet^N/N)) = \beta_\bullet;$$

$$\lim_{N\to\infty} v_N N \,\mathbb{E}(g(L_\bullet^N/N)) = \int_0^\infty g\nu_\bullet.$$

Tightness and identification of the limiting values to solutions of

$$F_t = F_0 - p_f \int_0^t F_s ds + \alpha_f \int_0^t F_s \wedge M_s \, ds + \sigma_f \int_0^t \sqrt{F_s \wedge M_s} dB_s^f + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le F_{s-} \wedge M_{s-}} h(z) \widetilde{N}^f + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le F_{s-} \wedge M_{s-}} (z - h(z)) N^f,$$

$$M_t = M_0 - p_m \int_0^t F_s ds + \alpha_m \int_0^t F_s \wedge M_s \, ds + \sigma_m \int_0^t \sqrt{F_s \wedge M_s} dB_s^m + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le F_{s-} \wedge M_{s-}} h(z) \widetilde{N}^m + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le F_{s-} \wedge M_{s-}} (z - h(z)) N^m$$

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genealogies ? infinite dimension ?

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&\int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le F_{s-} \wedge M_{s-}} h(z) \widetilde{N}^m + \int_0^t \int_{(0,\infty)^2} \mathbf{1}_{\theta \le F_{s-} \wedge M_{s-}} (z - h(z)) N^m
\end{aligned}
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