# Penalization of Galton-Watson processes

R. Abraham - P. Debs

Badajoz, WBPA18

# Notations

- $(Z_n)_{n\geq 0}$ Galton-Watson process starting from $Z_0 = 1$ under $\mathbb{P}$.
- Offspring distribution $q$.
- Mean $\mu$.
- Generating function of $\mu$: $f$.
- $\mathscr{F}_n = \sigma(Z_k, 0 \leq k \leq n)$.
- If $(M_n)_{n\in\mathbb{N}}$ is a non-negative martingale w.r.t. $(\mathscr{F}_n)_{n\in\mathbb{N}}$ with $M_0 = 1$, we define $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathscr{F}_n} = M_n.$$

# Super-critical GW conditioned on extinction



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Under $\mathbb{Q}$, $Z$ is a sub-critical offspring distribution with generating function

$$\tilde{f}(s) = \frac{f(\kappa s)}{\kappa}$$

and thus mean $\tilde{\mu} = f'(\kappa)$.
It is also the original GW process conditioned on extinction.

# The size-biased (sub)-critical GW process



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Under $\mathbb{Q}$, the genealogical tree of $Z$ is composed of an infinite spine decorated with copies of the original tree (Kesten 86).

It can be viewed as the GW process conditioned on non-extinction since

$$\mathbb{Q}(\cdot) = \lim_{n\to+\infty} \mathbb{P}(\cdot \mid Z_n > 0).$$

# Combination of the two previous martingales



Suppose $q$ super-critical with $q(0) > 0$.

$M_n = \frac{Z_n \kappa^{Z_n - 1}}{f'(\kappa)^n}$ is a martingale with $M_0 = 1$.

Under $\mathbb{Q}$, the genealogical tree is Kesten's tree of the associated sub-critical GW process.

# Penalization method (1)



Introduced by Roynette, Valois, Yor for Brownian motion.
Gives Brownian-like processes conditioned on specific zero-probability events.

Conditioning on a zero-probability event: for every $\Lambda_n \in \mathscr{F}_n$,

$$\lim_{m \to +\infty} \mathbb{P}(\Lambda_n | Z_{n+m} \in A) = \lim_{m \to +\infty} \frac{\mathbb{E}[\mathbf{1}_{\Lambda_n} \mathbf{1}_{Z_{n+m} \in A}]}{\mathbb{E}[\mathbf{1}_{Z_{n+m} \in A}]}$$

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Roynette et al's idea : replace the indicator function by another function $\varphi(Z_{m+n})$.

# Penalization method (2)

Let $\varphi$ be a real function.
If, for every $\Lambda_n \in \mathscr{F}_n$, we have

$$\lim_{m \to +\infty} \frac{\mathbb{E}\left[\varphi(Z_{n+m})\mathbf{1}_{\Lambda_n}\right]}{\mathbb{E}\left[\varphi(Z_{n+m})\right]} = \mathbb{E}\left[M_n \mathbf{1}_{\Lambda_n}\right]$$

then $M_n$ is a martingale with $M_0 = 1$.

# Example



If $\varphi(x) = x$,

$$\mathbb{E}[Z_{m+n} \mid \mathscr{F}_n] = Z_n \mu^m$$

and

$$\mathbb{E}[Z_{m+n}] = \mu^{m+n}$$

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Idea : take a function $\varphi$ of the form

$$\varphi(x) = H_p(x) s^x$$

with $s \in [0, 1]$ and

$$H_p(x) = \frac{1}{p!} x(x-1)\cdots(x-p-1).$$

# (Sub)-critical case



## Theorem

*Let $p \in \mathbb{N}^*$ and let $q$ be a critical or subcritical offspring distribution that admits a moment of order $p$ (and satisfies the $L \log L$ condition if $p = 1$).*
*Then, for every $s \in [0,1]$, we have, for every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathscr{F}_n$,*

$$\lim_{m \to +\infty} \frac{\mathbb{E}\left[H_p(Z_{m+n}) s^{Z_{m+n}} \mathbf{1}_{\Lambda_n}\right]}{\mathbb{E}\left[H_p(Z_{m+n}) s^{Z_{m+n}}\right]} = \mathbb{E}\left[Z_n/\mu^n \mathbf{1}_{\Lambda_n}\right]$$

# Super-critical case with $s \in [0,1)$



## Theorem

*Let $p \in \mathbb{N}^*$ and let $q$ be a super-critical offspring distribution that admits a moment of order $p$ (and satisfies the $L \log L$ condition if $p = 1$). let us set*

$$\mathfrak{a} = \min\{k \geq 0,\ q_k > 0\}.$$

*Then, for every $s \in [0,1)$, we have, for every $n \in \mathbb{N}$ and every $\Lambda_n \in \mathscr{F}_n$,*

$$\lim_{m \to +\infty} \frac{\mathbb{E}\left[H_p(Z_{m+n}) s^{Z_{m+n}} \mathbf{1}_{\Lambda_n}\right]}{\mathbb{E}\left[H_p(Z_{m+n}) s^{Z_{m+n}}\right]} = \begin{cases} \mathbb{E}\left[\frac{Z_n \kappa^{Z_n - 1}}{f'(\kappa)^n} \mathbf{1}_{\Lambda_n}\right] & \text{if } \mathfrak{a} = 0 \\ \mathbb{E}\left[q_1^{-n} \mathbf{1}_{Z_n = 1} \mathbf{1}_{\Lambda_n}\right] & \text{if } \mathfrak{a} = 1 \\ \mathbb{E}\left[q_{\mathfrak{a}}^{-\frac{\mathfrak{a}^n - 1}{\mathfrak{a} - 1}} \mathbf{1}_{Z_n = \mathfrak{a}^n} \mathbf{1}_{\Lambda_n}\right] & \text{if } \mathfrak{a} \geq 2 \end{cases}$$

# Special case

Only interesting case : $q$ super-critical and $s = 1$ or $s_n \to 1$.

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Rigth speed : $s_n = e^{-a/\mu^n}$ for some $a \geq 0$.

Recall that, under the $L \log L$ condition, $Z_n/\mu^n \to W$.
We set $\phi$ the Laplace transform of $W$.

# The limiting martingale



## Theorem

$$\lim_{m\to+\infty} \frac{\mathbb{E}\left[H_p(Z_{m+n})e^{-aZ_{m+n}/\mu^{m+n}}\mathbf{1}_{\Lambda_n}\right]}{\mathbb{E}\left[H_p(Z_{m+n})e^{-aZ_{m+n}/\mu^{m+n}}\right]} = \mathbb{E}\left[\frac{1}{\mu^{pn}} G_n^{(p)}(Z_n)\mathbf{1}_{\Lambda_n}\right]$$

*with*

$$G_n^{(p)}(x) = \frac{p!}{\phi^{(p)}(a)}\left(\sum_{i=1}^{p} a_i^{(p)}(n)H_i(x)\right)\phi(a/\mu^n)^x$$

*and*

$$a_i^{(p)}(n) = \phi(a/\mu^n)^{-i} \sum_{\substack{(n_1,\dots,n_i)\in(\mathbb{N}^*)^i \\ n_1+\cdots+n_i=p}} \prod_{r=1}^{i} \frac{\phi^{(n_r)}(a/\mu^n)}{n_r!}.$$

# The associated tree



The associated tree is a multitype inhomogeneous Galton-Watson tree such that

- The types run from 0 to $p$.
- The root is of type $p$.
- A node of type $\ell$ at level $n$ gives (independently of the other nodes) $k$ offspring with respective types $(\ell_1, \dots, \ell_k)$ with $\ell_1 + \cdots + \ell_k = \ell$ with probability

$$q_k \frac{1}{\mu^\ell} \frac{\ell!}{\phi^{(\ell)}(a/\mu^n)} \prod_{j=1}^{k} \frac{\phi^{(\ell_j)}(a/\mu^{n+1})}{\ell_j!}.$$

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- Nodes of type 1 give one node of type 1 and nodes of type 0 (Kesten tree).
- Nodes of type 2 give either one node of type 2 or two nodes of type 1, and nodes of type 0 $\longrightarrow$ a two-spine tree.