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Branching processes and stochastic equations

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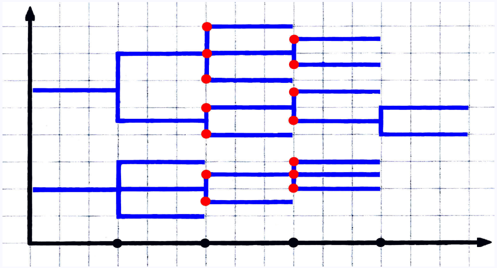
1. Galton-Watson branching processes

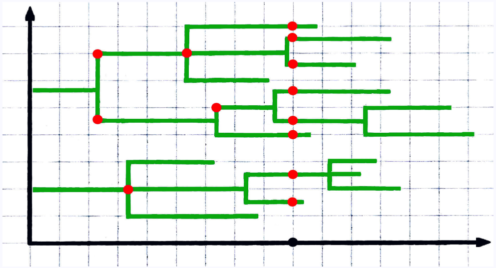
Let $\{\xi_{n,i}\}$ be a family of positive integer-valued i.i.d. random variables. Given X_0, we can define a Galton-Watson branching process by

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{n,i}, \qquad n \geq 1.$$

Problem A continuous-time/state model would be more realistic. However, the above formulation CANNOT be generalized directly to that setting.



discrete-time case



continuous-time case

2. Stochastic equations for CB-processes

Recall that a Galton-Watson branching process (GW-process) is defined by

$$X_k = \sum_{i=1}^{X_{k-1}} \xi_{k,i}, \qquad k \geq 1. \tag{1}$$

Suppose that $\mu := \mathbf{E}(\xi_{1,1}) < \infty$. Then $(1 - \mu = b)$

$$\begin{aligned} X_k &= X_{k-1} - (1-\mu)X_{k-1} + \sum_{i=1}^{X_{k-1}} (\xi_{k,i} - \mu), \\ X_n &= X_0 - \sum_{k=1}^{n} bX_{k-1} + \sum_{k=1}^{n} \sum_{i=1}^{X_{k-1}} (\xi_{k,i} - \mu). \end{aligned}$$

- A typical continuous-time/state branching process (CB-process) is defined by

$$x(t) = x(0) - \int_0^t bx(s-)ds + \int_0^t \int_0^{x(s-)} \int_0^\infty \xi \tilde{N}(ds, du, d\xi), \tag{2}$$

where $\tilde{N}(ds, du, d\xi) =$ compensated Poisson random measure on $(0, \infty)^3$; Bertoin/Le Gall ('06), Dawson/L ('06).

(1) for GW-process $\longleftrightarrow$ (2) for CB-process

Suppose that $\sigma \geq 0$ and b are constants, and $(z \wedge z^2)m(dz)$ is a finite measure on $(0, \infty)$. Let

- $W(ds, du)$ = Gaussian white noise on $(0, \infty)^2$ with intensity $dsdu$;
- $\tilde{N}(ds, dz, du)$ = compensated Poisson random measure $(0, \infty)^3$ with intensity $dsm(dz)du$.

Theorem 1 (Dawson/L '06; L/Ma '08) *There is a pathwise unique positive (strong) solution to [simple generalization of (2)]:*

$$x(t) = x(0) - b\int_0^t x(s)ds + \sigma \int_0^t \int_0^{x(s)} W(ds, du) + \int_0^t \int_0^\infty \int_0^{x(s-)} z\tilde{N}(ds, dz, du). \tag{3}$$

- The solution $\{x(t)\}$ to (3) is a general CB-process.

Example A Feller branching diffusion, the simplest CB-process, is defined by

$$x(t) = x(0) + \int_0^t \sqrt{x(s)}dB(s), \tag{4}$$

where $B(s)$ is a Brownian motion.

A Feller branching flow $\{X_t(v) : t \geq 0, v \geq 0\}$ can be defined by [special form of (3)]

$$X_t(v) = v + \int_0^t \int_0^{X_s(v)} W(ds, du). \tag{5}$$

The mapping $v \mapsto X_t(v)$ is non-decreasing.

● For $w \geq v \geq 0$, the process $\{X_t(w) - X_t(v) : t \geq 0\}$ is a Feller branching diffusion independent of $\{X_t(v) : t \geq 0\}$.

● Let $Y_t(dv)$ be the random measure on $\mathbb{R}_+$ such that $Y_t([0, v]) = X_t(v)$. Then $\{Y_t : t \geq 0\}$ is a special Dawson-Watanabe superprocess; Dawson/L ('12).

3. Laplace transforms of the CB-process

The branching mechanism of $\{x(t)\}$ is a function ϕ on $[0,\infty)$ defined by

$$\phi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(\mathrm{d}u). \tag{6}$$

The transition probabilities of $\{x(t)\}$ are characterized by ($\lambda, x \geq 0$)

$$\mathbf{E}_x\big[\exp\{-\lambda x(t)\}\big] = \exp\{-x v(t,\lambda)\}, \tag{7}$$

where $t \mapsto v(t,\lambda)$ is the unique solution of

$$\frac{\partial}{\partial t} v(t,\lambda) = -\phi(v(t,\lambda)), \quad v(0,\lambda) = \lambda. \tag{8}$$

Moreover, we have

$$\mathbf{E}_x\Big[\exp\Big\{-\theta\int_0^t x(s)ds\Big\}\Big] = \exp\{-x u(t,\theta)\}, \tag{9}$$

where $t \mapsto u(t,\theta)$ is the unique solution of

$$\frac{\partial}{\partial t} u(t,\theta) = \lambda - \phi(u(t,\theta)), \quad u(0,\theta) = 0. \tag{10}$$

4. Local and global maximal jumps

Example A special CB-process with jumps is defined by:

$$y(t) = x + \int_0^t \int_0^{y(s)} W(ds, du) + \int_0^t \int_0^\infty \int_0^{y(s-)} zN(ds, dz, du). \qquad (11)$$

We may also consider the Feller branching diffusion $\{x(t)\}$ defined by:

$$x(t) = x + \int_0^t \int_0^{x(s)} W(ds, du). \qquad (12)$$

By the independence of $\{x(t)\}$ and $\{N(ds, dz, du)\}$, we have

$$\begin{aligned} &\mathbf{P}_x(s \mapsto y(s) \text{ has no jumps on } (0, t]) \\ &\quad = \mathbf{P}_x\Big(\int_0^t \int_0^\infty \int_0^{y(s-)} zN(ds, dz, du) = 0, x(s) = y(s) \text{ for } 0 \le s \le t\Big) \\ &\quad = \mathbf{P}_x\Big(\int_0^t \int_0^\infty \int_0^{x(s-)} N(ds, dz, du) = 0\Big) \\ &\quad = \mathbf{P}_x\Big[\exp\Big\{-m(0, \infty)\int_0^t x(s)ds\Big\}\Big] \quad \text{(explicitly computable).} \qquad (13) \end{aligned}$$

Let $\Delta x(s) = x(s) - x(s-)$ and $\tau_r = \inf\{s > 0 : \Delta x(s) > r\}$.

Theorem 2 (He/L '15) *For any $r > 0$ we have (local maximal jump)*

$$\mathbf{P}_x\Big\{\max_{0<s\le t} \Delta x(s) \le r\Big\} = \mathbf{P}_x\{\tau_r > t\} = \exp\{-x u_r(t)\}, \tag{14}$$

where $t \mapsto u_r(t)$ is the unique solution of

$$\frac{\partial}{\partial t} u_r(t) = m(r,\infty) - \phi_r(u_r(t)), \quad u_r(0) = 0, \tag{15}$$

where

$$\phi_r(z) = \Big[b + \int_r^\infty u\, m(\mathrm{d}u)\Big] z + c z^2 + \int_0^r (\mathrm{e}^{-zu} - 1 + zu)\, m(\mathrm{d}u). \tag{16}$$

Theorem 3 (He/L '15) *Suppose that $\phi(z) \to \infty$ as $z \to \infty$. Then for any $r \ge 0$ with $m(r,\infty) > 0$ we have (global maximal jump)*

$$\mathbf{P}_x\Big\{\sup_{s>0} \Delta x(s) \le r\Big\} = \mathbf{P}_x\{\tau_r = \infty\} = \exp\{-x \phi_r^{-1}(m(r,\infty))\}. \tag{17}$$

These also gave distributional properties of the Lévy tree.

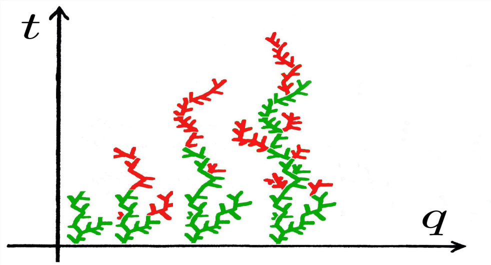
5. Tree-valued Markov processes

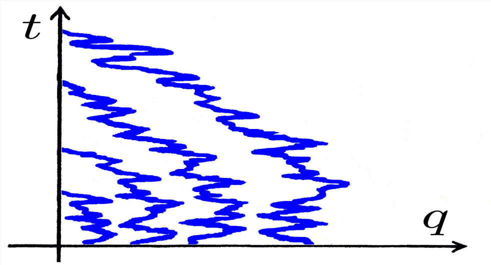
The study was initiated by Aldous/Pitman ('98).

From ϕ we can define a family of branching mechanisms (for some $\theta_0 \geq 0$):

$$\phi_q(\lambda) = \phi(\lambda - q) - \phi(-q), \qquad q \in T := [0, \theta_0] \text{ or } [0, \theta_0). \tag{18}$$

- Abraham/Delmas ('12): increasing tree-valued process $\{\mathscr{T}(q) : q \in T\}$.





- The tree-valued process and TWO path-valued processes:

$$\{\mathscr{T}(q) : q \in T\} \longleftrightarrow \{(X_t(q))_{t \geq 0} : q \in T\} \text{ or } \{X_t(q)_{q \in T} : t \geq 0\}. \tag{19}$$

6. Solution flow of a stochastic equation

The family $\{\phi_q : q \in T\}$, $T = [0, \theta_0]$ or $[0, \theta_0)$, can be expressed as:

$$\phi_q(\lambda) = b_q\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-z\lambda} - 1 + z\lambda)m_q(dz), \tag{20}$$

where $q \mapsto m_q(dz)$ is increasing and defines a measure $m(dq, dz)$ on $T \times (0, \infty)$. Let

- $W(ds, du) =$ Gaussian white noise based on $dsdu$;
- $\tilde{N}_0(ds, dy, dz, du) =$ compensated Poisson r.m. with intensity $dsm(dy, dz)du$.

Theorem 4 (L '14) *There is a pathwise unique positive strong solution flow to:*

$$\begin{aligned} X_t(q) \;&=\; X_0 - b_q\int_0^t X_s(q)ds + \sigma\int_0^t\int_0^{X_s(q)} W(ds, du) \\ &\quad + \int_0^t\int_{[0,q]}\int_0^\infty\int_0^{X_{s-}(q)} z\tilde{N}_0(ds, dy, dz, du). \end{aligned} \tag{21}$$

● This gives a construction of the path-valued processes $\{(X_t(q))_{t\geq 0} : q \in T\}$ and $\{X_t(q)_{q\in T} : t \geq 0\}$.

7. Structures of the two path-valued processes

● The path-valued processes $\{(X_t(q))_{t\geq 0} : q \in T\}$ and $\{X_t(q)_{q\in T} : t \geq 0\}$ are easier to handle than the tree-valued process $\{\mathscr{T}(q) : q \in T\}$.

Theorem 5 (L '14) *The path-valued process $\{(X_t(q))_{t\geq 0} : q \in T\}$ with state space $D^+[0,\infty)$ is a branching Markov process with inhomogeneous transition semigroup $(\mathbf{P}_{p,q} : q \geq p \in T)$ given by*

$$\int_{D^+[0,\infty)} e^{-\int_0^\infty f(s)w(s)ds} \mathbf{P}_{p,q}(\eta, dw) = \exp\Big\{-\int_0^\infty u_{p,q}(s,f)\eta(s)ds\Big\} \quad (22)$$

for $f \in C^+[0,\infty)$ with compact support, where

$$u_{p,q}(s,f) = f(s) + \phi_p(u_q(s,f)) - \phi_q(u_q(s,f)) \quad (23)$$

and $s \mapsto u_q(s) := u_q(s,f)$ is the unique compactly supported bounded positive solution to

$$u_q(s) + \int_s^\infty \phi_q(u_q(t))dt = \int_s^\infty f(t)dt, \qquad s \geq 0. \quad (24)$$

The increasing path $q \mapsto X_t(q)$ in $\{X_t(q)_{q\in T} : t \geq 0\}$ induces a measure Y_t on T, and $\{Y_t : t \geq 0\}$ is a process with state space $M(T) := \{\text{measures on } T\}$.

- Let $f \mapsto \Psi(\cdot, f)$ be the operator on $C^+(T)$ defined by

$$\Psi(x, f) = \int_T f(x \vee \theta)\beta_\theta d\theta + \int_T d\theta \int_0^\infty \big(1 - e^{-zf(x\vee\theta)}\big) n_\theta(dz). \tag{25}$$

Theorem 6 (L '14) *The measure-valued process $\{Y_t : t \geq 0\}$ is a Dawson-Watanabe superprocess with transition semigroup $(Q_t)_{t\geq 0}$ defined by*

$$\int_{M(T)} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = e^{-\langle \mu, V_t f\rangle}, \qquad f \in C^+(T), \tag{26}$$

where $t \mapsto V_t f$ is the unique locally bounded positive solution of

$$V_t f(x) = f(x) - \int_0^t [\phi_0(V_s f(x)) - \Psi(x, V_s f)] ds, \quad t \geq 0, x \in T. \tag{27}$$

- Thus $\{Y_t : t \geq 0\}$ has local branching mechanism $\phi = \phi_0$ and nonlocal branching mechanism Ψ; Chapter 2 of L ('11).

Summary: A natural way to define a continuous-time/state branching process $\{x(t) : t \geq 0\}$ is to use the stochastic integral equation

$$x(t) = x(0) - \int_0^t bx(s)ds + \sigma \int_0^t \int_0^{x(s)} W(ds, du) + \int_0^t \int_0^\infty \int_0^{x(s-)} z\tilde{N}(ds, dz, du).$$

Some variations of the equation have been used to study:

- Dawson/L (AOP '12): the flow of subordinators of Bertoin/Le Gall ('03, '05, '06);
- Xiong (AOP '13), He/L/Yang (SPA '14): Dawson–Watanabe superprocesses;
- L (AOP '14): the tree-valued processes of Abraham/Delmas ('12);
- L–Ma (ArXiv '13): estimation of the parameters of the CB-process;
- Pardoux/Wakolbinger (ArXiv '14): the logistic growth model of Lambert ('05);
- He–L (ArXiv '14): distribution of the maximal jump of the CB-process.

Some References

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Thank You!