Randomly biased random walks on trees

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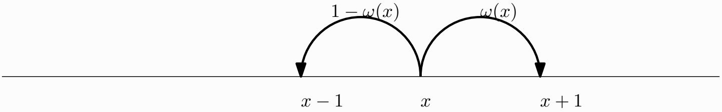
Overview

1. Introduction
   - Random walks in random environments on $\mathbb{Z}$
   - Randomly biased walks on trees
2. The recurrent case
3. Main result
   - Convergence in law for X_n
4. Proof
   - Localization of the biased walk
   - Approximation by invariant measure

Model of 1-d RWRE (S_n)

Let $\omega = \{\omega_x, x \in \mathbb{Z}\}$ be a family of i.i.d. random variables (and no constant) taking values in $(0, 1)$.

Given ω, $\{S_n, n \geq 0\}$ is a Markov chain on $\mathbb{Z}$ with probability transition :



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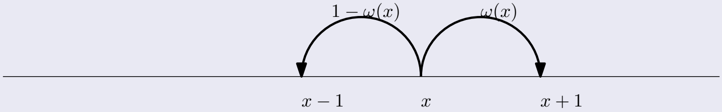

Asymptotic behaviors of (S_n)

References

- P. Révész : *Random walk in random and non-random environments* (1st edition : 1990, 2nd edition : 2005)
- O. Zeitouni : Lecture notes Saint Flour 2001.

Recurrence/transience criteria : Solomon (1975)

- (S_n) is recurrent if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) = 0$;
- $S_n \to \infty$ if and only if $\mathbb{E}(\log \frac{1-\omega_x}{\omega_x}) < 0$.

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How big is (S_n) ?

Transient case : Kesten, Kozlov and Spiter (1976)

when $S_n \to \infty$, $\frac{S_n}{n^\varrho}$ converges in law, with $\varrho \in (0, 1]$.

Recurrent case : Sinai (1982)'s localization

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RWRE on trees : Lyons and Pemantle (1992)

Random environments

Let $\mathbb{T}$ be a regular tree (or more generally a supercritical Galton-Watson tree), rooted at $\varnothing$. Let $\omega = \{(\omega(x,y), y \in \mathbb{T})_{x \in \mathbb{T}}\}$ be a family of random variables such that $\sum_{y \in \mathbb{T}: y \sim x} \omega(x,y) = 1$, $\omega(x,y) > 0$ iff $x \sim y$ ($x \sim y$ means x and y are adjacent).

Random walk in random environment (X_n) on a tree :

Conditioned on ω, (X_n) is a Markov chain taking values in $\mathbb{T}$ with

$$\mathbb{P}_\omega\Big(X_{n+1} = y \,\Big|\, X_n = x\Big) = \omega(x,y), \qquad \forall x,\, y \in \mathbb{T}.$$

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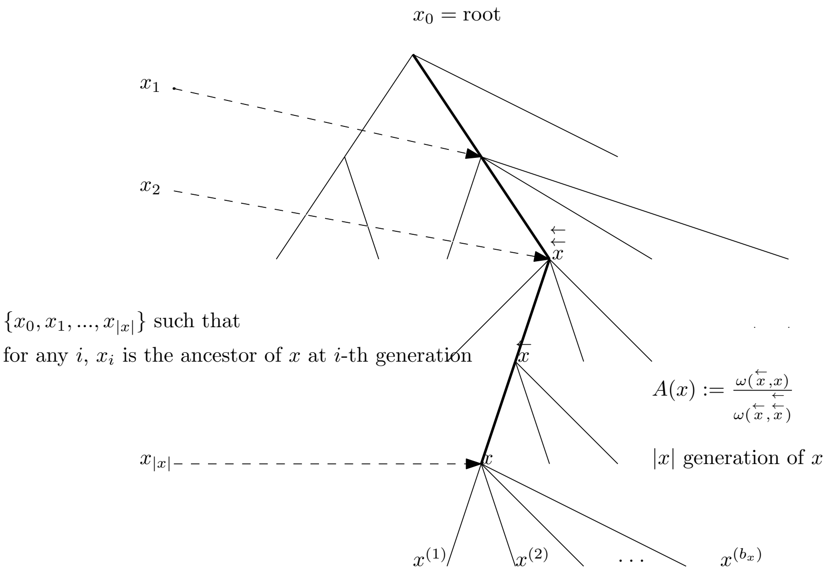
Notations in a Galton-Watson tree $\mathbb{T}$, rooted at $\varnothing$



Notations

Instead of looking at $\omega(x, \cdot)$, it is more convenient to use the notation $A(x) := \omega(\overleftarrow{x}, x)/\omega(\overleftarrow{x}, \overleftarrow{\overleftarrow{x}})$ [referred as the biase].

When all $A(x) = \lambda$ some positive constant, the walk is called λ-biased walk on a Galton-Watson tree [Lyons, Pemantle and Peres (1995, 1996)].

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Recurrence/transience criteria

Hypothesis :

We assume that for all $|x| \geq 2$, $\{A(x^{(1)}), \dots, A(x^{(\mathtt{b}_x)})\}$ are i.i.d. and distributed as the vector $\{A_1, \dots, A_{\mathtt{b}}\}$, where $\mathtt{b}_x$ denotes the number of children of x. Define

$$\psi(t) := \log \mathbb{E}\Big(\sum_{i=1}^{\mathtt{b}} A_i^t \Big), \qquad \forall t \in \mathbb{R}.$$

Lyons and Pemantle (1992)'s theorem :

1. if $\inf_{0\leq t\leq 1}\psi(t)>0$, then RWRE (X_n) is a.s. transient.
2. If $\inf_{0\leq t\leq 1}\psi(t)=0$, then RWRE (X_n) is a.s. recurrent.
3. If $\inf_{0\leq t\leq 1}\psi(t)<0$, then (X_n) is a.s. positive recurrent.

Subdiffusive case

Theorem (H. and Shi 2007)

If $\inf_{0 \le t \le 1} \psi(t) = 0$ and $\psi'(1) < 0$, then almost surely,

$$\max_{0 \le i \le n} |X_i| = n^{\nu + o(1)},$$

where

$$\nu := 1 - \max(\frac{1}{2}, \frac{1}{\kappa}) \in (0, \frac{1}{2}],$$

and

$$\kappa := \inf\{t > 1 : \psi(t) = 0\} \in (1, \infty].$$

Slow movement case

Theorem (Faraud, H. and Shi 2012)

If $\psi(1) = \psi'(1) = 0$, then almost surely,

$$\lim_{n\to\infty} \frac{1}{(\log n)^3} \max_{0\le i\le n} |X_i| = \frac{8}{3\pi^2\psi''(1)}.$$

Remark (G. Faraud)

If $\inf_{0\le t\le 1} \psi(t) = 0$ and $\psi'(1) > 0$, then (X_n) is positive recurrent.

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- (sub)diffusive case ($\kappa > 2$) : Faraud (2013) proved CLT for $\kappa > 5$; E. Aïdékon and Loïc de Raphélis (convergence to Brownian tree) for $\kappa > 2$.
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Critical case : $\psi(1) = \psi'(1) = 0$

When $\psi(1) = \psi'(1) = 0$, the associated potential process V is a branching random walk in the "boundary case", where

$$V(x) := - \sum_{y \in]\!]\varnothing, x]\!]} \log A(y), \qquad \forall x \in \mathbb{T}.$$

Question

Recall that $\max_{0 \le k \le n} |X_k| \sim c(\log n)^3$ a.s. What is the renormalization for $|X_n|$?

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Main result

Theorem (H. and Shi'15+)

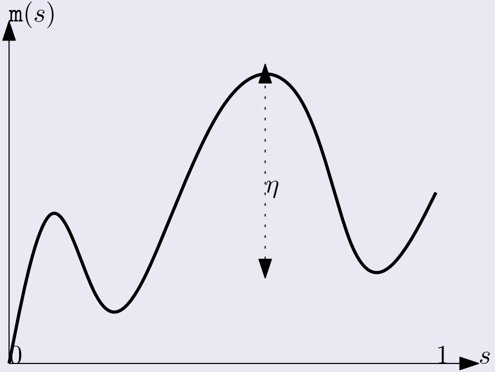
Under $\mathbb{P}(\,\cdot\,|\mathbb{T}=\infty)$,

$$\frac{\sigma^2\,|X_n|}{(\log n)^2} \xrightarrow{(d)} \mathbb{X}_\infty,$$

with $\mathbb{P}(\mathbb{X}_\infty \in db)/db = \sqrt{\frac{1}{2\pi\, b}}\,\mathbb{P}(\eta \le \frac{1}{\sqrt{b}})$, and $\sigma^2 = \psi''(1)$.

Definition of η

η is the maximal drawdown of the Brownian meander $\mathtt{m}$



Maximal drawdown

Proposition

$\int_0^\infty \sqrt{\frac{1}{2\pi\, b}}\, \mathbb{P}(\eta \leq \frac{1}{\sqrt{b}})\, db = 1.$

Proof of Proposition by Marc Yor (2013)

This is equivalent to show that $\mathbb{E}\left[\frac{1}{\eta}\right] = \sqrt{\frac{\pi}{2}}$, which can be done by using the stochastic calculus...

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Finitely-dimensional convergence

Remark

For any $\kappa \geq 1$ and $0 < t_1 < t_2 < ... < t_\kappa \leq 1$,

$$\left(\frac{|X_{[t_i n]}|}{(\log n)^2}, 1 \leq i \leq \kappa\right)$$

are asymptotically independent and converge to the same limiting law.

Sinaï's localization : when X takes values in $\mathbb{Z}$.

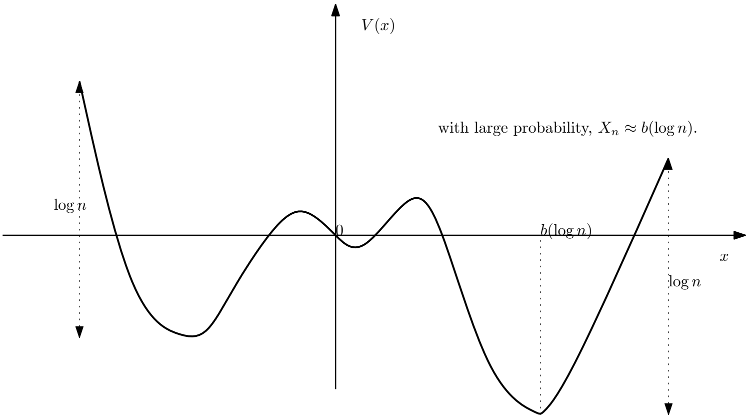
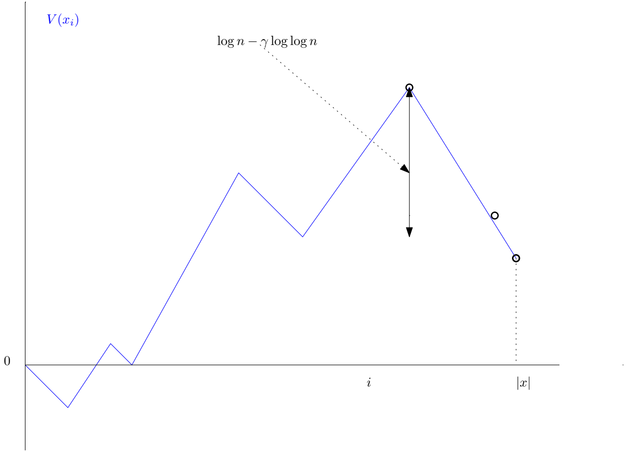


FIGURE: Sinaï's valley, where V denotes the potential process of X.

Picture of the reflecting barrier $\mathscr{L}_n^{(\gamma)}$

$$x \in \mathscr{L}_n^{(\gamma)} \text{ is roughly equivalent to } V^{\#}(x) > \log n - \gamma \log\log n$$

but none of the ancestors of x does, where
$V^{\#}(x) := \max_{\varnothing < z \le y \le x}(V(z) - V(y))$.



Reflecting barriers

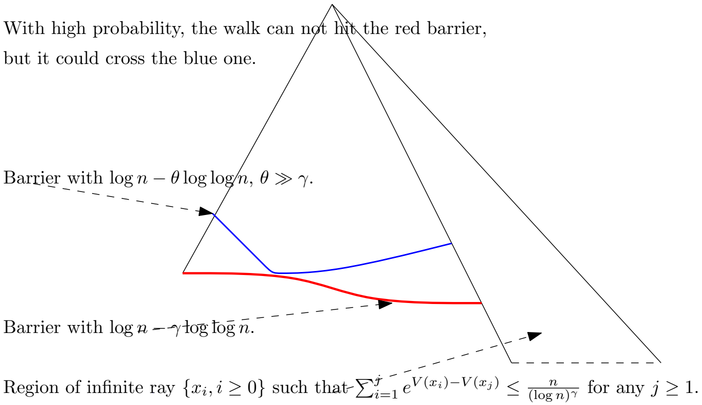


FIGURE: Two reflecting barriers $\mathscr{L}_n^{(\gamma)}$ and $\mathscr{L}_n^{(\theta)}$, $\gamma < 2$ and θ large

Invariant probability measure

- Let $\pi_n(\cdot)$ be the invariant probability measure of the biased walk reflected at $\mathscr{L}_n^{(\gamma)}$:

$$\pi_n(x) \approx \frac{1}{Z_n} \mathrm{e}^{-V(x)}, \qquad x \le \mathscr{L}_n^{(\gamma)}.$$

- Main step : Show that (modulo the parity of n)

$$\text{the law of } X_n \approx \pi_n(\cdot).$$

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Proof of Theorem : by assuming the approximation of invariant measure

$$
\begin{aligned}
P_\omega\left(a \le \frac{|X_n|}{(\log n)^2} \le b\right) &\approx \sum_{a(\log n)^2 \le |x| \le b(\log n)^2} \pi_n(x) \\
&\approx \frac{1}{Z_n} \sum_{a(\log n)^2 \le |x| \le b(\log n)^2} 1_{(x < \mathscr{L}_n^{(\gamma)})} \mathrm{e}^{-V(x)},
\end{aligned}
$$

with $Z_n = \sum_{x \le \mathscr{L}_n^{(\gamma)}} \mathrm{e}^{-V(x)}$.

Proof of Theorem : continuation

A functional limit theorem by Th. Madaule (2013+)

Choose $|x| = n$ according to the probability propositional to $\mathrm{e}^{-V(x)}$. The linear interpolation of $(n^{-1/2} V(x_{[tn]}), 0 \le t \le 1)$ converges in law to $(\sigma\, \mathtt{m}_t, 0 \le t \le 1)$ with $\mathtt{m}$. the Brownian meander.

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Study the behaviors of Z_n etc...

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THANK YOU !