Stochastic Particle Systems on Non-Homogeneous Spatial Lattice Structures

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Outline

Introduction

We consider a model of stochastic lattice systems with the following key features:

- their elements can move on the lattice;
- they have a finite number of "sources" on the lattice, where the elements can produce new copies or disappear;
- the behaviour of all elements, being independent of each other, is covered by the same stochastic law.

An important example of such stochastic multicomponent lattice systems is continuous-time branching processes with particles walking on the lattice $\mathbf{Z}^{\mathbf{d}}$.

Introduction

The branching processes with particles walking on $\mathbf{Z}^{\mathbf{d}}$ are usually called *branching random walks* (BRW).

Recent investigations have demonstrated that continuous-time BRW on $\mathbf{Z}^{\mathbf{d}}$ give an important example of stochastic processes whose evolution depend on

- **the structure of an environment**,
- **the spatial dynamics**.

It is convenient to describe such processes in terms of birth, death, and walks of particles on a lattice.

The structure of an environment is defined by the offspring reproduction law at a finite number of generation centers situated on the lattice.

The spatial dynamics of particles is considered under different assumptions about underlying random walks:

- symmetric or non-symmetric,
- with or without the finite variance of jumps.

Introduction

Informal Description of BRW on $\mathbf{Z^d}$

- The population of individuals is initiated at time $t = 0$ by a single particle at a point $x \in \mathbf{Z^d}$.
- Being outside of the sources the particle performs a continuous time random walk on $\mathbf{Z^d}$ until reaching one of the sources.
- At a source it spends an exponentially distributed time and then either jumps to a point $y \in \mathbf{Z^d}$ (distinct from the source) or dies producing just before the death a random number of offsprings.
- The newborn particles behave independently and stochastically in the same way as the parent individual.

▸ BRW with several sources

Introduction

Objects of Study

We will be mainly interested in describing the evolution of particles on $\mathbf{Z}^{\mathbf{d}}$ in terms of

- **the number of particles $\boldsymbol{n(t,x,y)}$ at a point $\boldsymbol{y} \in \mathbf{Z}^{\mathbf{d}}$**
- their **moments**

$$m_k(t,x,y) := E_x n^k(t,x,y), \quad k \in \mathbf{N},$$

  where E_x denotes the mathematical expectation under the condition $n(0,x,y) = \delta_y(x)$.

Previous Studies

Mainly concentrated on the study of the limit behavior of the process **$\boldsymbol{n(t,x,y)}$ under fixed spatial coordinates**.

Our Goal

To investigate the limit behavior of $\boldsymbol{n(t,x,y)}$ when **both coordinates**, $\boldsymbol{t}$ and $\boldsymbol{y}$, may vary, that is to undertake **the spatio-temporal analysis** of the evolution of the system.

Spatial Configuration of the Sources

Limitation in Supercritical BRW Investigations

The presence of positive eigenvalues in the spectrum of evolutionary operator implies an exponential growth of the number of particles in an arbitrary lattice point and on the entire lattice. Therefore, in the previous studies the authors were usually **limited** to finding only **the leading eigenvalue**.

Progress in Supercritical BRW Investigations

At the same time for the spatio-temporal analysis, the information about whether the positive eigenvalue is **unique**, or if it is **not unique** then how it is **located** with respect to other eigenvalues, can be significant in the analysis of the behavior of BRWs.

Results

In connection with this, it was found that the number of positive eigenvalues of the discrete spectrum of the evolutionary operator and their multiplicity depend not only on **the intensity of the sources** but on **the spatial configuration of the sources**.

Formal Description of BRW

Random Walk

Let $A=(a(x,y))_{x,y\in\mathbf{Z}^{\mathbf{d}}}$ be the matrix of transition intensities of a random walk:

- $a(x,y)\geq 0$ for $x\neq y$, $a(x,x)<0$;
- $a(x,y)=a(y,x)=a(0,y-x)=a(y-x)$ and $\sum\limits_{z}a(z)=0$;
- for every $z\in\mathbf{Z}^{\mathbf{d}}$ there exists a set of vectors $z_1,z_2,\ldots,z_k\in\mathbf{Z}^{\mathbf{d}}$ such that $z=\sum_{i=1}^{k}z_i$ and $a(z_i)\neq 0$ for $i=1,2,\ldots,k$;

Branching in Sources

- $f(u):=\sum\limits_{n=0}^{\infty}b_nu^n$, where $b_n\geq 0$ for $n\neq 1$, $b_1<0$ and $\sum\limits_{n}b_n=0$.
- $\beta_r:=f^{(r)}(1)<\infty$, $r\in\mathbf{N}$, and $\beta:=\beta_1$.

Evolutionary Operator for Sources of Equal Intencity

In the BRW models with finitely many sources, there arise multipoint perturbations of the symmetric random walk generator $\mathscr{A}$, which have the form

$$\mathscr{H}_\beta = \mathscr{A} + \beta \sum_{i=1}^{N} \Delta_{x_i},$$

where $x_i \in \mathbb{Z}^d$, $\mathscr{A} : l^p(\mathbb{Z}^d) \to l^p(\mathbb{Z}^d)$, $p \in [1,\infty]$, is a symmetric operator acting by formula

$$(\mathscr{A} u)(z) := \sum_{z' \in \mathbf{Z^d}} a(z-z')u(z'),$$

$\Delta_x = \delta_x \delta_x^T$, $\delta_x = \delta_x(\cdot)$ denotes the column vector on the lattice which is equal to 1 at the point x and to zero at the other points.

The perturbation $\beta \sum_{i=1}^{N} \Delta_{x_i}$ of the operator $\mathscr{A}$ can result in the appearance of positive eigenvalues of the operator $\mathscr{H}_\beta$, and the multiplicity of each eigenvalue does not exceed **the number of terms** in the last sum.

BRW with a Few Sources of Branching

The discrete spectrum σ_d consists of no more than N nonnegative eigenvalues provided that there are N sources of the branching on the lattice [Yarovaya, 2012].

▸ BRW with a Few Sources of Branching

The Green's Function

Resolvent of the Operator $\mathscr{A}$

The structure of the eigenvalues and eigenfunctions of $\mathscr{H}_\beta$ are closely related to the transition probabilities $p(t,x,y)=p(t,0,y-x)=p(t,0,x-y)$ of the underlying random walk. They satisfy the Kolmogorov backward equation

$$\partial_t p = \mathscr{A} p, \quad p(0,x,y)=\delta_y(x).$$

The Green's function for them is as follows:

$$G_\lambda(x,y) := \int\limits_0^\infty e^{-\lambda t} p(t,x,y)\,dt = \frac{1}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{e^{i(\theta,x-y)}}{\lambda - \widehat{\mathscr{A}}(\theta)}\,d\theta, \quad \lambda \geq 0,$$

where $\widehat{\mathscr{A}}$ is the Fourier transform of the operator $\mathscr{A}$.

Analysis of BRWs depends on whether the value of $G_0 = G_0(0,0)$ is **finite** or **infinite**.

Definition

A random walk is *transient* if $G_0(0,0)<\infty$ and *recurrent* if $G_0(0,0)=\infty$.

Assumptions on Variance of Random Walk Jumps

In what follows we will consider $a(\cdot)$ under two different assumptions:

1. $\sum\limits_{z} |z|^2 a(z) < \infty$, where $|z|$ is Euclidean norm of a vector z.
2. $a(z) \sim \dfrac{H\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}$, $\alpha \in (0,2)$, where $H(\cdot)$ is continuous positive and symmetric on the sphere $\mathbb{S}^{d-1} = \{z \in \mathbb{R}^d : |z| = 1\}$ function.

In the **first** case $G_0 = \infty$ for $d = 1, 2$ and $G_0 < \infty$ for $d \geq 3$.

In the **second** case $\sum\limits_{z} |z|^2 a(z) = \infty$ which implies infinite variance of jumps.
In this case $G_0 = \infty$ for $d = 1$ and $\alpha \in [1, 2)$, while $G_0 < \infty$ for $d = 1$ and $\alpha \in (0, 1)$ or $d \geq 2$ and $\alpha \in (0, 2)$.

Discrete Spectrum of Evolutionary Operator

Let β_c denotes the minimal value of the source intensity such that for $\beta > \beta_c$ the spectrum of $\mathscr{H}_\beta$ has positive eigenvalues.

Theorem

If $G_0 = \infty$ then $\beta_c = 0$ for $N \geq 1$. If $G_0 < \infty$ then $\beta_c = G_0^{-1}$ for $N = 1$, and $0 < \beta_c < G_0^{-1}$ for $N \geq 2$.

For example, when $G_0 < \infty$ and $N = 2$ the quantity β_c is computed in [Yarovaya, 2012]:

$$\beta_c = (G_0 + \widetilde{G}_0)^{-1},$$

where $\widetilde{G}_0 = G_0(x_1, x_2)$.

The next theorem gives an additional information about the structure of the spectrum of the operator $\mathscr{H}_\beta$.

Theorem

Let $N \geq 2$. Then for $\beta > \beta_c$ the operator $\mathscr{H}_\beta$ may have no more than N positive eigenvalues of finite multiplicity

$$\lambda_0(\beta) > \lambda_1(\beta) \geq \cdots \geq \lambda_{N-1}(\beta) > 0,$$

where the eigenvalue $\lambda_0(\beta)$ has multiplicity one. Besides, there exists a value $\beta_{c_1} > \beta_c$ such that for $\beta \in (\beta_c, \beta_{c_1})$ the operator has no other eigenvalues except $\lambda_0(\beta)$.

Discrete Spectrum of Evolutionary Operator

In general, the problem of finding the eigenvalues of an operator is complicated. To help solving it one may use the following assertion proved in [Yarovaya, 2012] for a more general case of different types of branching sources.

Theorem

An eigenvalue λ belongs to the discrete spectrum of the operator $\mathscr{H}_\beta$ iff the following system of linear equations

$$V_i - \beta \sum_{j=1}^{N} G_\lambda(x_i, x_j) V_j = 0, \quad i = 1, \dots, N,$$

with respect to variables $\{V_i\}_{i=1}^N$ has at least one nontrivial solution.

Example

Let $\mathscr{A} = \kappa\Delta$, $\kappa > 0$, be the lattice Laplacian, $N \geq 2$, and the points $x_1, \dots, x_N$, at which sources of equal intensities are positioned, form the vertices of a regular simplex. Such a kind of simplices in $\mathbb{Z}^d$ can be obtained, for example, as an arbitrary combination of the standard basis vectors.

Example for N=3

Then the critical values β_c and β_{c_1} can be computed explicitly:

$$\beta_c = (G_0 + (N-1)\widetilde{G}_0)^{-1}, \quad \beta_{c_1} = (G_0 - \widetilde{G}_0)^{-1},$$

where $\widetilde{G}_0 = G_0(x_i, x_j)$ for $i \neq j$ (in our case all the values $G_0(x_i, x_j)$ for different $i \neq j$ coincide with each other and then the value $\widetilde{G}_0$ does not depend on i and j).

Remark

The operator $\mathscr{A}$ should not necessarily be equal to $\kappa\Delta$. In order that the assertion of Example remained to be valid, it suffices to require that the values of the Fourier transform $\widehat{\mathscr{A}}(\theta)$ of the intensity function $a(z)$ do not change under any permutation of coordinates of the vector $\theta = \{\theta_1, \theta_2, \dots, \theta_d\}$. The latter property will take place, for example, if the function $a(z)$ does not change its values under any permutation of coordinates of the vector $z = \{z_1, z_2, \dots, z_d\}$.

Definition

Let there exist $\varepsilon_0 > 0$ such that for $\beta \in (\beta_c, \beta_c + \varepsilon_0)$ the operator $\mathscr{H}_\beta$ has only one (accounting multiplicity) positive eigenvalue $\lambda(\beta)$ satisfying $\lambda(\beta) \to 0$ for $\beta \downarrow \beta_c$. Then the supercritical BRW will be called *weakly supercritical for β close to β_c*.

Question

Whether any supercritical BRW for $\beta > \beta_c$ sufficiently close to β_c is weakly supercritical?

Theorem

Every supercritical BRW for $\beta \downarrow \beta_c$ is weakly supercritical.

Weakly Supercritical BRWs with Finite Variance of Jumps

Using the asymptotic behavior for $p(t,0,0) \sim \gamma_d t^{-\frac{d}{2}}$, as $t \to \infty$, we obtain asymptotic behavior of G_λ under assumption 1.

Theorem

If $\lambda \to 0$ then

$$G_\lambda = \begin{cases} \widetilde{\gamma}_1 (\sqrt{\lambda})^{-1} \cdot (1 + o(1)), & d = 1, \\ -\widetilde{\gamma}_2 \ln \lambda \cdot (1 + o(1)), & d = 2, \\ G_0 - \widetilde{\gamma}_3 \sqrt{\lambda} \cdot (1 + o(1)), & d = 3, \\ G_0 + \widetilde{\gamma}_4 \lambda \ln \lambda \cdot (1 + o(1)), & d = 4, \\ G_0 - \widetilde{\gamma}_d \lambda \cdot (1 + o(1)), & d \geq 5, \end{cases}$$

where $\widetilde{\gamma}_d$ is a positive constant.

Weakly Supercritical BRWs with Finite Variance of Jumps

From the previous theorem we get the asymptotic behavior of $\lambda_0(\beta)$, as $\beta \to \beta_c$.

Theorem (Molchanov, Yarovaya, 2012)

The eigenvalue $\lambda_0(\beta)$ of the operator $\mathscr{H}_\beta$ has the following asymptotic behavior as $\beta \to \beta_c$:

$$\begin{aligned}
\lambda_0(\beta) &\sim c_1\beta^2, && d=1,\\
\lambda_0(\beta) &\sim e^{-c_2/\beta}, && d=2,\\
\lambda_0(\beta) &\sim c_3(\beta-\beta_c)^2, && d=3,\\
\lambda_0(\beta) &\sim c_4(\beta-\beta_c)\ln^{-1}((\beta-\beta_c)^{-1}), && d=4,\\
\lambda_0(\beta) &\sim c_d(\beta-\beta_c), && d\ge 5,
\end{aligned}$$

where c_i, $i \in \mathbf{N}$, is a positive constant.

Transition probabilities for heavy tailed BRW

Under **appropriate regularity conditions** on the tails of the jump distributions, asymptotic behavior of the transition probability $p(t,0,x)$ uniformly in x, t, $|x|+t\to\infty$ is investigated in [Agbor, Molchanov & Vainberg, 2014].

From these results for fixed spatial coordinates, the next asymptotic relation immediately follows:

$$p(t,x,y)\sim C_{d,\alpha}t^{-\frac{d}{\alpha}},\quad t\to\infty,\quad 0<\alpha<2.$$

We get the local limit theorem for $p(t,x,y)$ **in the absence of any regularity conditions** by using the multidimensional analog (Rytova, Yarovaya, 2014) of the known Watson's Lemma.

▸ Watson's Lemma

Using the asymptotic representation $p(t,0,0) \sim C_{d,\alpha} t^{-\frac{d}{\alpha}}$, as $t \to \infty$, we obtain asymptotic behavior of G_λ under Assumption 2.

Theorem (Yarovaya, 2014)

If $\lambda \to 0$ *then*

$$G_\lambda = \begin{cases} \breve{\gamma}_{1,\alpha} \lambda^{\frac{1-\alpha}{\alpha}} \cdot (1+o(1)), & d=1, \quad 1<\alpha<2, \\ -\breve{\gamma}_{1,\alpha} \ln \lambda \cdot (1+o(1)), & d=1, \quad \alpha=1, \\ G_0 - \breve{\gamma}_{1,\alpha} \sqrt{\lambda} \cdot (1+o(1)), & d=1, \quad 0<\alpha<1, \\ G_0 - \breve{\gamma}_{d,\alpha} \lambda \cdot (1+o(1)), & d \geq 2, \quad 0<\alpha<2, \end{cases}$$

where $\breve{\gamma}_{i,\alpha}$, $i \in \mathbf{N}$, *is a positive constant for every* α.

From the previous theorem we get the asymptotic behavior of $\lambda_0(\beta)$, as $\beta \to \beta_c$, for BRW under Assumption 2.

Theorem (Yarovaya, 2014)

The eigenvalue $\lambda_0(\beta)$ of the operator $\mathscr{H}_\beta$ has the following asymptotic behavior as $\beta \to \beta_c$:

$$
\begin{aligned}
&\lambda_0(\beta) \sim c_{1,\alpha}\beta^{\frac{\alpha}{\alpha-1}}, && d=1, \quad 1<\alpha<2,\\
&\lambda_0(\beta) \sim e^{-c_{1,1}/\beta}, && d=1, \quad \alpha=1,\\
&\lambda_0(\beta) \sim c_{1,\alpha}(\beta-\beta_c), && d=1, \quad 0<\alpha<1,\\
&\lambda_0(\beta) \sim c_{d,\alpha}(\beta-\beta_c), && d\ge 2, \quad 0<\alpha<2.
\end{aligned}
$$

where $c_{i,\alpha}$, $i \in \mathbf{N}$, is a positive constant for every α.

For spatio-temporal analysis of particle systems we apply the methods of the spectral theory of operators with multipoint perturbations. Such perturbations plays an important role in the intermittency theory for the so-called parabolic Anderson localization problem [Gärtner & Molchanov, 1990, Gärtner, König & Molchanov, 2007], where Anderson localization is a general wave phenomenon that applies to the transport of electromagnetic waves, acoustic waves, quantum waves, spin waves, etc.

We use resolvent analysis of a bounded symmetric operator with multi-points potential to study the distribution of population inside the front of propagation of the weakly supercritical BRW on $\mathbf{Z}^d$.

Spectral Analysis Approach

In [Cranston, Koralov, Molchanov & Vainberg, 2009]
an approach based on the resolvent analysis of evolutionary operators has been proposed to study a continuous model of homopolymers on $\mathbf{R}^d$ with path large deviations for Brownian motion.

Drawback: this model does not cover the case of BRW on $\mathbf{Z}^d$.

Large Deviations for BRWs

Topics to be discussed

1. Limit theorems for the Green's functions of transition probabilities.
2. The case when the spectrum of the evolution operator of mean numbers of particles contains only one positive isolated eigenvalue.
3. Properties of the front of population and the structure of the population inside of the front and near its boundary.

Additional Assumptions on Random Walk Generator

- $\sum\limits_{z\neq 0} a(z) = -a(0) = 1$, where $a(z) \geq 0$ for $z \neq 0$ (normalization);
- $H(\theta) = \sum_{z\neq 0} a(z)(e^{(\theta,z)} - 1) = \sum_z a(z)\cosh(\theta,z) < \infty$, $\theta \in [-\pi,\pi]^d$.

The last condition is essential for the large deviation theory, in the case of slow decay of $a(\cdot)$ the theory is completely different. Under this condition the function $a(z)$ obeys the estimation

$$|a(z)| \leq c_1 e^{-c_2|z|^\gamma}, \quad z \in \mathbf{Z}^d, \quad c_1, c_2 > 0, \quad \gamma > 1.$$

The BRW Model

Behaviour at Sources

At the moment τ_1 of the first reaction, the particle is duplicated

$$P \to P + P$$

and both copies start moving independently with the same law from the point $x(\tau_1)$. The rate of duplication $\beta V(x) \geq 0$, $x \in \mathbf{Z}^{\mathbf{d}}$, is called *the potential.*

For simplicity, we exclude death of particles and more complex transformations like $P \to P + P + P$, etc.

We will concentrate on the case when $\beta V(x)$ has a finite support:

$$V(x) = \sum_{l=1}^{N} V_l \delta_{x_l}(x), \quad V_l > 0.$$

The BRW Model

Generating Functions

The finite-dimensional distribution of the field $n(t,x,\cdot)$ can be expressed in terms of the generating function

$$u_z(t,x,y_1,y_2,\ldots,y_m) = u_{z_1,z_2,\ldots,z_m}(t,x,y_1,y_2,\ldots,y_m) = \mathbf{E}_x z_1^{n(t,x,y_1)} \cdots z_m^{n(t,x,y_m)}$$

where $y_1,\ldots,y_m \in \mathbf{Z}^d$, $|z_j| \le 1$. For the function u_z, the Kolmogorov backward equation takes place

$$\partial_t u_z = \mathscr{A}_x u_z + \beta V(x)(u_z^2 - u_z), \tag{1}$$

where

$$u_z(0,x,y_1,\ldots,y_m) = \begin{cases} z_i, & x = y_i,\ i = 1,\ldots,m, \\ 1, & x \ne y_1,\ldots,y_m. \end{cases}$$

The Kolmogorov backward equation generates the system of equations for the moment functions. For example, if $m_1(t,x,y) = \mathbf{E}_x n(t,x,y)$ then

$$\begin{cases} \partial_t m_1 & = \mathscr{H}_\beta m_1, \\ m_1(0,x,y) & = \delta_y(x). \end{cases}$$

The BRW Model

Moment Equations. Hamiltonian $\mathscr{H}_\beta$

The Hamiltonian $\mathscr{H}_\beta = \mathscr{A} + \beta V(\cdot) I$ is a bounded self-adjoint operator on $l^2(\mathbf{Z}^d)$. Similarly we can obtain the equations for the high-order moments. For the second product moment

$$m_2(t,x,y_1,y_2) = \mathbf{E}_x n(t,x,y_1) n(t,x,y_2), \quad y_1 \neq y_2,$$

the equation takes the form:

$$\begin{cases} \partial_t m_2(t,x,y_1,y_2) & = \mathscr{H}_\beta m_2 + 2\beta V(x) m_1(t,x,y_1) m_1(t,x,y_2), \\ m_2(0,x,y_1,y_2) & = \delta(x-y_1) + \delta(x-y_2). \end{cases}$$

All the moment equations contain the basic operator $\mathscr{H}_\beta$ and the spectral properties of $\mathscr{H}_\beta$ play a key role in the further analysis of the particle field $n(t,x,\Gamma)$ on a set $\Gamma \subseteq \mathbf{Z}^d$.

The joint spatio-temporal asymptotics of the transition probabilities

For the investigation of BRWs with large deviations, particularly of their Green's functions, it is urgent to know asymptotic behavior of the transition probabilities in the situation when ***the spatial and temporal variables grow jointly***.

In the study we obtain a formula for such a joint asymptotics. Based on the obtained formula we give a scale of changes of transition probabilities for the random walk under joint growth of time and the spatial variable having the power growth in time.

Main Results

Global Theorem (Molchanov, Yarovaya 2013)

For a fixed C, as $t \to \infty$, uniformly over $|x| \le Ct$, where $|\cdot|$ is Euclidean norm of a vector,

$$p(t,0,x) \sim \frac{e^{t\left[H\left(\lambda_*\left(\frac{x}{t}\right)\right)-\left(\lambda_*\left(\frac{x}{t}\right),\frac{x}{t}\right)\right]}}{(2\pi t)^{d/2}\sqrt{\det B\left(\lambda_*\left(\frac{x}{t}\right)\right)}} = \frac{e^{-tH_*\left(\frac{|x|}{t}\right)}}{(2\pi t)^{d/2}\sqrt{\det B\left(\lambda_*\left(\frac{|x|}{t}\right)\right)}},$$

where $p(t,0,x) = P_0(x(t) = x)$, $\lambda_*\left(\frac{x}{t}\right)$ is a single root of the equation $\nabla H(\lambda_*) = \frac{x}{t}$ and

$$B(\lambda) = \left[\frac{\partial^2 H}{\partial x_i \partial x_j}(\lambda)\right] = \operatorname{Hess} H(\lambda).$$

Main Results

Put $h_\mu(\theta) = \left(\lambda_*\left(\frac{\theta}{s_*}\right), \theta\right)$, where s_* is the solution of the equation $\mu = H\left(\lambda_*\left(\frac{\theta}{s}\right)\right)$.

Large Deviation Theorem for Green's Function(Molchanov, Yarovaya 2013)

For fixed $\mu > 0$ and $|x| \to \infty$ we have the following asymptotic representation

$$G_\mu(0,x) \sim \frac{c_d\, e^{-|x| h_\mu(\theta)}}{|x|^{\frac{d-1}{2}} \sqrt{s_*^d} \sqrt{\det B(\lambda_*(\theta/s_*))}},$$

where $\theta = \frac{x}{|x|}$, and c_d is a positive constant depending on the dimension d of the lattice.

Therefore, for any $\mu > 0$ we have

$$\psi_0(x,\beta) \asymp G_\mu(0,x) \asymp \frac{e^{-|x|h_\mu\left(\frac{x}{|x|}\right)}}{|x|^{\frac{d-1}{2}}}.$$

Here, $f(x) \asymp g(x)$ for $0 < c \le \frac{f}{g} \le C < \infty$. The representation $h_\mu(x/|x|) = h_\mu(\theta)$ has been obtained in the Large Deviation Theorem for the Green's function.

If $\beta \to \beta_c$ then $\lambda_0(\beta) \to 0$, and the last expression may be represented in the explicit form:

$$\psi_0(x,\beta) \asymp G_{\lambda_0(\beta)}(0,x) \asymp e^{-\sqrt{\lambda_0(\beta)}|x|(1+o(1))}.$$

Concept of the Population Front

Let $x = x_0 = 0$ and $m_1(t,0,y) = \mathbf{E}_0 n(t,0,y)$. Assume that $\beta_c < \beta < \beta_1$, i.e. $\sigma_d(\mathscr{H}_\beta)$ contains a single eigenvalue $\lambda_0(\beta) > 0$. Then

$$m_1(t,0,y) = e^{\lambda_0(\beta)t}\psi_0(y)\psi_0(0) + O(1),$$

where $\|\psi_0\|_2 = 1$.

Assume that $n(0,0,y) = \delta_0(y)$ and $\beta_c < \beta < \beta_{c1}$. Let us call

$$\Gamma_t = \{y : m_1(t,0,y) \le C\}$$

the population front.

Propagation of the Population

The definition of the front depends on a constant C, but with the logarithmical accuracy it will not depend on C, and instead of C one can consider any function $o(e^{\varepsilon t})$.

Theorem

Let $\ln G_\mu(0,y) \sim -|y| h_\mu\left(\frac{y}{|y|}\right)$, *as* $|y| \to \infty$. *Then*

$$\Gamma_t = \left\{ y : \frac{|y|}{t} h_1\left(\lambda_0(\beta), \frac{y}{|y|}\right) \geq \lambda_0(\beta) + o(1) \right\},$$

where $o(1)$ *is a function tending to zero under the joint unbounded growth of* $|y|$ *and* t *subjected to the condition* $|y| = O(t)$.

For $\beta \to \beta_c$ the front has approximately spherical form:

$$\Gamma_t \approx \left\{ y : |y| \geq t\sqrt{\lambda_0(\beta)} \right\}.$$

As in the classical Kolmogorov - Petrovskii - Piskunov (KPP) model, the population is spreading linearly in time.

Remark (Heavy Tailed BRWs)

For $\beta \to \beta_c$ *the front of particles propagates exponentially fast in time.*

Case of Low Dimensions

Now we will formulate the results about the structure of the population inside the front for some special cases.

The total number of particles at a moment t satisfies the relation $n(t) = \sum_{y\in\mathbf{Z}^d} n(t,0,y)$ if the initial particle starts from $x=0$ at the moment $t=0$.

The following theorem gives the description of the population inside the front and near to its boundary for $d=1$ and $d=2$.

Theorem

Let $x\in\mathbf{Z}^d$, $d=1$ or $d=2$. If for some $c_1,c_2>0$ we have $c_1 t\le|x|\le c_2 t$, as $t\to\infty$, and $\mathbf{E}_0\, n(t,0,x) = m_1(t,0,x)\to\infty$ then

$$\frac{n(t,0,x)}{\mathbf{E}_0\, n(t,0,x)} \xrightarrow{law} n_\infty,$$

where the distribution of n_∞ is independent on x and obeyed the relation $P_0\{n_\infty>0\}=1$. Moreover,

$$\frac{n(t)}{\mathbf{E}_0\, n(t)} \xrightarrow{law} n_\star.$$

Case of High Dimensions

For $d \geq 3$ the situation is more complicated.

Main objects of investigation of the particle population structure inside a front are limit probabilities

$$\pi_k(x) = \lim_{t\to\infty} \mathbf{P}_x\{n(t) = k\}, \quad k = 1, 2, \ldots \tag{2}$$

and their generating functions

$$\phi(z, x) = \sum_{k=1}^{\infty} \pi_k(x) z^k = \lim_{t\to\infty} \mathbf{E}_x z^{n(t)}. \tag{3}$$

If $|z| < 1$ then the last limit is equal to zero for $n(t) \to \infty$. When the population is bounded, i.e. $\limsup_{t\to\infty} n(t) < \infty$, and $n(\infty)$ is a limit number of particles then $\phi(z, x) = \mathbf{E}_x z^{n(\infty)}$.

The special role play the basic generating functions:

$$\phi_i(z) = \phi(z, x_i) = \mathbf{E}_{x_i} z^{n(\infty)}.$$

Case of High Dimensions

If $\beta \le \beta_c$ then $\varphi_i(z)|_{z=1} = 1$, and the population remains bounded for $t \to \infty$. There is the stabilization of $n(t)$ for $t \ge t_0(\omega)$.

If $\beta > \beta_c$ then $\phi_i(z)|_{z=1} < 1$, and population is growing exponentially (with probability $1 - \phi_i(1)$ if it starts from x_i at the moment $t = 0$) or it remains bounded and

$$\mathbf{P}_{x_i}\{n(t) = k\} \longrightarrow \pi_k(x_i) = k!\phi_i^{(k)}(1).$$

The system of equations for the basic generating functions can be solved explicitly for several particular symmetric configurations.

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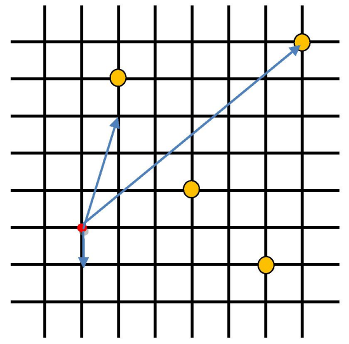
BRW on $\mathbf{Z}^2$ at the initial time t=0

Example



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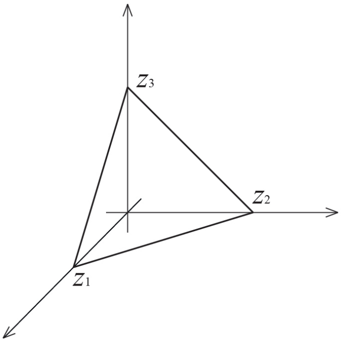
Example for N=3



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Watson's Lemma

Watson's Lemma

Let $\alpha > 0$, $f(x) \in (C[a,b])$ and $f(0) \neq 0$. Then

$$\int\limits_0^a f(x) e^{-tx^\alpha}\, dx \sim \frac{f(0)}{\alpha} \Gamma\left(\frac{1}{\alpha}\right) t^{-\frac{1}{\alpha}}, \quad t \to \infty.$$

Multidimensional Watson's Lemma

Let $\alpha > 0$, $f(\cdot) \in C([-\pi,\pi]^d)$, $f(0) \neq 0$, $S(\cdot) \in C([-\pi,\pi]^d)$, $S(x) \sim \eta\left(\frac{x}{\|x\|}\right) \|x\|^\alpha$, as $x \to 0$, and for a function $\eta(\cdot) \in C(\mathbf{S}^{d-1})$ satisfies the enqualities:

$$0 < \eta_* \leq \eta(u) \leq \eta^* < \infty, \quad u \in \mathbf{S}^{d-1}.$$

Then

$$\int\limits_{[-\pi,\pi]^d} f(x) e^{-tS(x)}\, dx \sim 2\pi^{\frac{d}{2}} \Gamma^{-1}\left(\frac{d}{2}\right) \frac{f(0)}{\alpha} \Gamma\left(\frac{d}{\alpha}\right) (\eta_0 t)^{-\frac{d}{\alpha}}, \quad t \to \infty,$$

where $\eta_0 \in [\eta_*, \eta^*]$.

Return

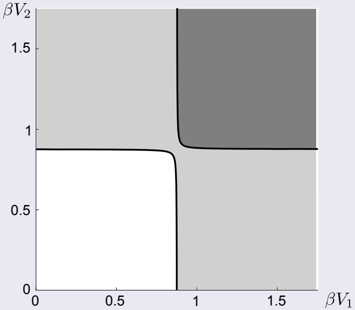
Spectrum of $\mathscr{H}_\beta$

Example. Application to BRW with Two Sources

Let

$$\mathscr{H}_\beta = \Delta + \beta V_1 \delta_{x_1} + \beta V_2 \delta_{x_2}$$

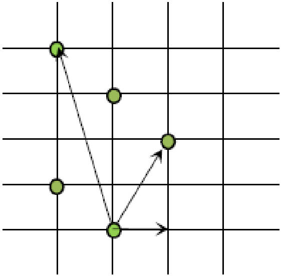
The set of parameters $\{\beta V_1, \beta V_2\}$ for which the operator $\mathscr{H}_\beta$ has one positive eigenvalue (the light grey area) and two positive eigenvalues (dark grey area)



Hence trespassing the line separating the white and light-grey areas corresponds to the weakly supercritical case.

BRW with a Few Sources of Branching

$$A+\beta_1\Delta_0+\beta_2\Delta_0+\beta_3\Delta_0+\beta_4\Delta_0+\beta_5\Delta_0$$



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