Sevastyanov Branching Processes with Non-homogeneous Poisson Immigration

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History - 1

- The class of age-dependent branching processes with dependence of the individual life span τ and the individual offspring ν was proposed and studied by Sevastyanov (1964, 1968, 1968a).
- In fact the Sevastyanov process generalizes the age-dependent model of Bellman and Harris (1947, 1952) in which the evolution of each individual is also described by the vector (τ, ν), but in this case the random variables τ and ν are independent.

- The first model of branching processes with immigration was introduced and investigated by Sevastyanov (1957) in the continuous-time Markov case and when the times of immigration form a homogeneous Poisson process.
- The discrete time case (BGW process with immigration) was first considered by Heathcote (1965, 1966) and many others (see books by Athreya and Ney (1972), Jagers(1975), and Assmusen and Hering (1983)).
- Bellman-Harris branching processes allowing time-homogeneous immigration were investigated by Jagers (1968), Pakes (1972), Radcliffe (1972), Pakes and Kaplan (1974)).
- Sevastyanov branching processes with homogeneous Poisson immigration were considered by Yanev(1972).

Motivation - by Applications

Recently, age-dependent branching processes with offspring distribution $h(s) = p + qs + rs^2$ and immigration occurring according to an inhomogeneous Poisson process have been considered to describe the evolution of cell populations.

- *Yakovlev A.Y., Yanev N.M.* Branching stochastic processes with immigration in analysis of renewing cell populations// Mathematical Biosciences 2006. V. 203. P. 37-63.
- *Yakovlev A.Y., Yanev N.M.* Age and residual lifetime distributions for branching processes// Statistics and Probability Letters 2007. V. 77. P. 503-513.
- *Hyrien O., Yanev N.M.* Age-dependent branching processes with non-homogeneous Poisson immigration as models of cell kinetics//Eds: D. Oakes, W.J. Hall, and A. Almudevar, "Modeling and Inference in Biomedical Sciences: In Memory of Andrei Yakovlev"
- *Hyrien, O., Peslak, S.A., Yanev, N.M., Palis, J.* (2014). Stochastic modeling of stress erythropoiesis using a two-type age-dependent branching process with immigration. J. Math. Biol., in press.

Motivation - by Theory

For many years, we have studied several, related classes of branching processes with migration, independent immigration and state-dependent immigration.

- *N. M. Yanev and K. V. Mitov* Critical Branching Processes with Nonhomogeneous Migration// Ann. Probab. 1985. V. 13(3). P. 923-933.
- *Mitov K.V., Vatutin V.A., Yanev N.M.* Continuous-time branching processes with decreasing state-dependent immigration// Adv. Appl. Prob. 1984. V. 16. P. 697-714.
- *Mitov K.V., Yanev N.M.* Bellman-Harris branching processes with state-dependent immigration// J. Appl. Prob. 1985. V. 22. P. 757-765.

Sevastyanov branching process with non-homogeneous Poisson immigration

The results discussed in the talk follow the classical scheme:

- Asymptotic behavior of the moments.
- Asymptotic behavior of the probability for visiting state 0.
- Limit theorems.

The subcritical, critical and supercritical cases are investigated for various intensity of immigration.

Definitions: Sevastyanov branching process

Basic assumptions (in the absence of immigration):

- Each particle lives for a random amount of time τ and at the end of its lifespan it produces ν new particles. The life-length τ and the number of children ν are dependent.
- Particles (individuals, cells) evolve independently of each other.
- The process $Z(t), t \geq 0$, is the total number of particles alive at time t.
- At time $t = 0$, the process starts with one new particle, $Z(0) = 1$.

Notations - 1

- $G(t) = \Pr\{\tau \le t\},\ t \ge 0$, lifespan d.f.
- The joint distribution of the particle's evolution:

$$\mathbf{P}\{\tau \in B, \nu = n\} = \int_B p_n(u)dG(u),$$

for every Borel set $B \subset \mathbf{R}$.
- Offspring p.g.f. $h(u,s) = \sum_{n=0}^{\infty} p_n(u)s^n,\ h(u,1) = 1$.
- P.g.f. $F(t,s) = \mathbf{E}[s^{Z(t)}|Z(0) = 1],\quad t \ge 0,\quad s \in [0,1]$, satisfies the following non-linear integral equation

$$F(t,s) = s(1 - G(t)) + \int_0^t h(u, F(t-u,s))dG(u)$$

with initial condition $F(0,s) = s$.

Notations - 2

- Denote the first three factorial moments of the offspring of a particle of age u by

$$a(u) = h'_s(u,1), \quad b(u) = h''_{ss}(u,1), \quad c(u) = h'''_{sss}(u,1).$$

- Define the p.g.f.

$$h(s) = \int_0^\infty h(u,s)dG(u), \quad |s| \leq 1$$

of the offspring of a particle during its entire lifespan.

- Denote also

$$a = h'(1), \quad b = h''(1), \quad c = h'''(1).$$

Criticality

Note that the Malthusian parameter α is determined as usually from the equation

$$\int_0^\infty e^{-\alpha x} a(x) dG(x) = 1.$$

In the subcritical case $a < 1$ we assume that α exists, in which case $\alpha < 0$. A Sevastyanov branching process is said to be subcritical if $a < 1(\alpha < 0)$, critical if $a = 1, b > 0(\alpha = 0)$, or supercritical if $a > 1(\alpha > 0)$. In the talk we discuss all cases.
Remark. In the case $a = 1, b = 0$, the branching process $Z(t) \equiv 1$ and the total progeny is an ordinary renewal process..

Non-homogeneous Poisson Immigration-1

$(S_k, I_k), k = 0, 1, 2, \ldots$ are i.i.d. random vectors, where:

$$0 = S_0 < S_1 < S_2 < S_3 < \ldots$$

are the jump points of a non-homogeneous Poisson process $\xi(t)$,

$\{I_k\}$ are i.i.d. non-negative integer valued.

At every jump-point S_n a random number I_n, $n = 0, 1, 2, \ldots$ independent identically distributed Sevastyanov branching processes start.

Sevastyanov branching process with non-homogeneous Poisson Immigration

The process $\{Y(t),\ t \geq 0\}$ is called Sevastyanov branching process with non-homogeneous Poisson immigration (SBPNPI):

$$Y(t) = \sum_{k=1}^{\xi(t)} \sum_{j=1}^{I_k} Z^{(j)}(t - S_k) \ \text{if}\ \xi(t) > 0; \quad Y(t) = 0 \ \text{if}\ \xi(t) = 0,$$

where $\{Z^{(j)}(t)\}$ are independent Sevastyanov branching processes governed by the same pair $(G(t), h(t,s))$.

Notations and equations

Let $r(t)$ be the intensity of $\xi(t)$ and $R(t) = \int_0^t r(u)du$, then

$$\mathbf{P}\{\xi(t) = n\} = \frac{e^{-R(t)}R^n(t)}{n!}, \quad = 0, 1, 2, ...$$

Denote by

$$g(s) = \mathbf{E}[s^{I_n}] = \sum_{k=0}^{\infty} q_k s^k$$

the p.g.f. of the immigrants.

Denote the p.g.f. of $Y(t)$, by

$$\Phi(t;s) = \mathbf{E}\{s^{Y(t)}|Y(0) = 0\}, \Phi(0;s) = 1.$$

Then the following equation holds:

$$\Phi(t;s) = \exp\left\{-\int_0^t r(t-u)(1 - g(F(u;s)))du\right\}$$

Equations for the joint p.g.f.

Introduce the p.g.f.
$\Phi(t,\tau;s_1,s_2) = \mathbf{E}[s_1^{Y(t)} s_2^{Y(t+\tau)} | Y(0)=0], \tau \geq 0$. Then

$$\begin{aligned}\Phi(t,\tau;s_1,s_2) &= \exp\{-\int_0^t r(u)[1-g(F(t-u,\tau;s_1,s_2))]du \\ &- \int_t^{t+\tau} r(v)[1-g(F(t,\tau-v;1,s_2))]dv\},.\end{aligned}$$

where

$$\begin{aligned}F(t,\tau;s_1,s_2) &= \int_0^t h(F(t-u,\tau;s_1,s_2))]du \\ &+s_1 \int_t^{t+\tau} h(v;F(t+\tau-v;s_2))dv\} \\ &+s_1 s_2[1-G(t+\tau)].\end{aligned}$$

Basic conditions

We study the process $\{Y(t), t \geq 0\}$ under the following basic conditions:
$m_I = g'(1-) < \infty, \;\; b_I = g''(1-) < \infty$;
the individual moments $a = h'(1)$ and $b = h''(1)$ are finite;
the lifetime d.f. $G(t)$ is not lattice with a finite mean

$$M = \mathbf{E}[\tau] = \int_0^\infty u dG(u) < \infty.$$

Introduce

$$\begin{aligned} M_1(t) &= \mathbf{E}[Z(t)], M_2(t) = \mathbf{E}[Z(t)(Z(t)-1)], W(t) = \mathbf{Var}[Z(t)], \\ A(t) &= \mathbf{E}[Y(t)], B(t) = \mathbf{E}[Y(t)(Y(t)-1)], V(t) = \mathbf{Var}[Y(t)]. \end{aligned} \quad (1)$$

Subcritical case $a < 1 (\alpha < 0)$

Theorem 1. Let $r(t) \sim re^{\rho t}$ with $r > 0$.
(A) If $\rho < 0$ then
$\lim_{t\to\infty} P\{Y(t) = k | Y(t) > 0\} = q_k > 0, k = 1, 2, \dots .$
(B) If $\rho > 0$ then
(i) LLN: $\zeta(t) = Y(t)/A(t) \to 1$, *a.s.* and L_2 as $t \to \infty$;
(ii) CLT: $X(t) = [Y(t) - A(t)]/\sqrt{V(t)} \to N(0, \sigma^2)$ *in distribution as* $t \to \infty$, where

$$0 < \sigma^2 = \frac{\int_0^\infty e^{-\rho u} \left(\gamma M_2(u) + \gamma_2 M_1(t)^2(u)\right) du}{\int_0^\infty e^{-\rho u} \left(\gamma M_2(u) + \gamma M_1(t)(u) + \gamma_2 M_1(t)^2(u)\right) du} < \infty.$$

Theorem 2. Let $r(t) \sim rt^{\theta}$ with $r > 0$.
(A) If $\theta < 0$ then
$\lim_{t\to\infty} P\{Y(t) = k | Y(t) > 0\} = q_k > 0, k = 1, 2, \dots$, where

$$\Psi^*(s) = \sum_{k=1}^{\infty} q_k s^k = 1 - \frac{\int_0^{\infty}(1 - g(F(u,s)))du}{\int_0^{\infty}(1 - g(F(u,0)))du}, \quad 0 \le s \le 1.$$

(B) If $\theta > 0$ then as $t \to \infty$
(i) LLN: $\zeta(t) = Y(t)/A(t) \to 1$, in L_2. The convergence is **almost surely** if $\theta > 1$.
(ii) CLT: $X(t) = [Y(t) - A(t)]/\sqrt{V(t)} \to N(0, \sigma^2)$ *in distribution as* $t \to \infty$, where

$$0 < \sigma^2 = 1 - \frac{m_I \int_0^{\infty} M_1(t)du}{\int_0^{\infty}[m_I M_2(t)(u) + (m_I + \ b_I)M_1(t)^2(u)]du} < \infty.$$

Subcritical case $a < 1 (\alpha < 0)$

Theorem 3. Let $\lim_{t\to\infty} r(t) = r > 0$. Then there exists a limiting distribution

$$\lim_{t\to\infty} P\{Y(t) = k\} = Q_k > 0, k = 0, 1, 2, \dots,$$

where

$$\Psi^*(s) = \sum_{k=0}^{\infty} Q_k s^k = \exp\{-r \int_0^\infty [1 - g(F(u,s))] du\}, |s| \le 1.$$

Corollary. If $G(t) = 1 - e^{-t/M}, t \ge 0$, then $\{Z(t), t \ge 0\}$ will be a Markov branching process and then

$$\Psi^*(s) = \exp\{-r \int_s^1 \frac{1 - g(x)}{f(x)} dx\},$$

where $f(s) = (h(s) - s)/M$, $M = \mathbf{E}[\tau] = \int_0^\infty u dG(u) < \infty$. The similar result was obtained by Sevastyanov (1957).

Critical case: Basic conditions

The intensity of the Poisson process satisfies one of the following condition:

$$r(t) \downarrow 0, \int_0^\infty r(t)dt = R \in (0, \infty),$$

$$r(t) = \frac{1}{t+1},$$

$$r(t) \sim t^\delta L_R(t)\ , t \to \infty, \delta \in (-1, 0],$$

$$r(t) \uparrow r > 0, \quad t \to \infty,$$

$$r(t) \sim t^\delta L_R(t)\ , t \to \infty, \delta > 0,$$

where $L_R(t)$ are some s.v.f. as $t \to \infty$.

Critical case: Moments - I

Denote by

$$A(t) = \mathbf{E}[Y(t)] \text{ and } B(t) = \mathbf{E}[Y(t)(Y(t)-1)].$$

Since $a = \int_0^\infty a(u)dG(u) = 1$, therefore

$$G_a(t) = \int_0^t a(u)dG(u), \text{ for } t \geq 0$$

is a proper distribution function on $[0,\infty)$. Denote by

$$M_a = \int_0^\infty ua(u)dG(u) = \int_0^\infty udG_a(u) \text{ the first moment of } G_a(t).$$

We need the following asymptotic formulas ($M = \mathbf{E}[\tau]$):

- $$M_1(t) = \mathbf{E}[Z(t)] = \frac{M}{M_a} + o\left(\frac{1}{t}\right), \quad t \to \infty.$$

  (Sevastyanov (1971), §VIII.8, Theorem 6).

- $$M_2(t) = \mathbf{E}[Z(t)(Z(t)-1)] = \frac{M^2}{M_a^3} bt + B_1 + o(1), \quad t \to \infty,$$

  where B_1 is a constant. (Sevastyanov (1971), §VIII.8, Theorem 13).

Critical case: Asymptotic of the mean $A(t) = \mathbf{E}[Y(t)]$

Proposition 1. Assume the conditions of (Sevastyanov (1971), §VIII.8, Theorem 6) hold true. Then as $t \to \infty$:

$$A(t) \sim m_I \frac{M}{M_a} R(t), \quad t \to \infty.$$

Critical case: Asymptotic of the mean $A(t) = \mathbf{E}[Y(t)]$

Depending on the rate of change of $r(t)$, the asymptotic of $A(t)$ is as follows:

$$r(t) \downarrow 0, \int_0^\infty r(t)dt = R \in (0,\infty) \quad \Rightarrow \quad A(t) \to m_I \frac{M}{M_a} R.$$

$$r(t) = \frac{1}{t+1} \quad \Rightarrow \quad A(t) \sim m_I \frac{M}{M_a} \log t.$$

$$r(t) \sim t^\delta L_R(t), \delta \in (-1, 0] \quad \Rightarrow \quad A(t) \sim m_I \frac{M}{M_a} \frac{t^{1+\delta}}{1+\delta} L_R(t).$$

$$r(t) \uparrow r > 0, \quad \Rightarrow \quad A(t) \sim m_I \frac{M}{M_a} rt.$$

$$r(t) \sim t^\delta L_R(t), \delta > 0 \quad \Rightarrow \quad A(t) \sim m_I \frac{M}{M_a} \frac{t^{1+\delta}}{1+\delta} L_R(t).$$

Critical case: Asymptotic of $B(t) = \mathbf{E}[Y(t)(Y(t)-1)]$

Proposition 2. Assume the conditions of (Sevastyanov (1971), §VIII.8, Theorem 13) hold true. If $r(t) \sim t^{\delta} L_R(t)$ for $\delta \geq 0$ then as $t \to \infty$:

$$B(t) \sim m_I \int_0^t r(t-u) M_2(u) du \sim \frac{m_I M^2 b}{M_a^3(\delta+2)} R(t) t.$$

Critical case: Probability for non extinction - I

We study the asymptotic behavior of

$$\begin{aligned} D(t) &:= \Pr\{Y(t) > 0\} \\ &= 1 - \Phi(t, 0) = 1 - \exp\left(-\int_0^t r(t-u)Q(u)du\right). \end{aligned}$$

We assume that the conditions of (Sevastyanov (1971), §IX.1, Theorem 1) hold.
Depending on the behavior of $r(t)$ the asymptotic of the probability $\Pr\{Y(t) > 0\}$ is as follows:

Critical case: Probability for non extinction - II

Proposition 3. If $r(t) \downarrow 0, \int_0^\infty r(t)dt = R \in (0, \infty)$ holds and there exists a function $k(t)$ such that

$$k(t) \to \infty, \quad k(t)/t \to 0, \; r(k(t)) = o\left(\frac{1}{t \log t}\right), t \to \infty,$$

then

$$\Pr\{Y(t) > 0\} = \frac{2Rm_l M_a}{bt}(1 + o(1)), \quad t \to \infty.$$

2 If $r(t) = \dfrac{1}{t+1}$ holds then

$$\Pr\{Y(t) > 0\} = \frac{4M_a m_l \log t}{bt}(1 + o(1)), \quad t \to \infty.$$

Critical case: Probability for non extinction - III

3 If $r(t) \sim t^{\delta} L_R(t), \delta \in (-1, 0]$ holds then

$$\Pr\{Y(t) > 0\} = \frac{2M_a m_I r(t) \log t}{b}(1 + o(1)), \quad t \to \infty.$$

4 If $r(t) \uparrow r > 0$, holds then

$$\Pr\{Y(t) > 0\} \to 1, \quad t \to \infty.$$

5 If $r(t) \sim t^{\delta} L_R(t), \delta > 0$ holds then

$$\Pr\{Y(t) > 0\} \to 1, \quad t \to \infty.$$

Critical case: Limit theorem 4, $r(t) = o(1/t)$

Theorem 4. If $r(t) \downarrow 0$, $\int_0^\infty r(t)dt = R \in (0, \infty)$ and there exists a function $k(t)$ such that

$$k(t) \to \infty, \;\; k(t)/t \to 0, \; r(k(t)) = o\left(\frac{1}{t \log t}\right), t \to \infty$$

then

$$\lim_{t\to\infty} \Pr\{Y(t)D(t) \leq x | Y(t) > 0\} = 1 - e^{-\frac{m_I MR}{Ma}x}, x \geq 0.$$

The limiting distribution of the theorem corresponds to that obtained by Yaglom (1947).

Critical case: Limit theorem 5, $r(t) = 1/(t+1)$

Theorem 5. If $r(t) = \dfrac{1}{t+1}$ then

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\lambda Y(t) t^{-x}} | Y(t) > 0\right] = \frac{x}{2}, \quad x \in [0,1].$$

Corollary. Under the conditions of the theorem

$$\lim_{t\to\infty} \mathbf{P}\{\log Y(t)/\log t \le x | Y(t) > 0\} = \frac{x}{2}\mathbf{1}_{\{0\le x\le 1\}} + \frac{1}{2}\mathbf{1}_{\{x\ge 1\}},$$

for $x \ge 0$.

Theorem 6. If $r(t) = \dfrac{1}{t+1}$ then

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\lambda Y(t)D(t)/\log t} | Y(t) > 0\right] = \frac{1}{2} + \frac{1}{2(1+\frac{m_I M}{M_a}\lambda)}.$$

Corollary. Under the conditions of the theorem

$$\lim_{t\to\infty} \mathbf{P}\{Y(t)D(t)/\log t \le x | Y(t) > 0\} = \frac{1}{2}+\frac{1}{2}(1-\exp(-\frac{M_a}{Mm_I}x)),$$

for $x \geq 0$.

Critical case: $r(t) = 1/(t+1)$. Comments

Under the condition $r(t) = 1/(1+t)$ using different normalization we obtained two singular to each other limiting distributions. From the above theorems and their corollaries, we see that the non-degenerate sample paths are of two types:
- a half of them grows up linearly with slope an exponentially distributed random variable:
$Y(t) \sim \xi t$ *(in distribution)*, $\xi \in Exp$.
- the logarithm the other half of them grows up as $\log t$ with coefficient an uniformly on the unit interval random variable:
$Y(t) \sim t^{\eta}$ *(in distribution)*, $\eta \in U(0,1)$.

Theorem 7. Assume $r(t) \sim t^{\delta} L_R(t), \delta \in (-1, 0]$. Then

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\lambda Y(t) t^{-x}}\right] = x, \quad x \in [0, 1].$$

Corollary. Under the conditions of the theorem

$$\lim_{t\to\infty} \Pr\left\{\frac{\log Y(t)}{\log t} \le x | Y(t) > 0\right\} = x, \quad x \in [0, 1].$$

Critical case: Limit theorem 8

Theorem 8. Assume $r(t) \uparrow r > 0$. Then

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\lambda Y(t)/R(t)}\right] = \left(1 + \frac{Mb}{2M_a^2 r}\lambda\right)^{-2m_l M_a/b}, \quad \lambda > 0.$$

Corollary. Under the conditions of the theorem the limiting distribution is Gamma with parameters $\theta = \frac{Mb}{2M_a^2 r}$ and $\varkappa = \frac{2m_l M_a}{b}$, that is

$$\lim_{t\to\infty} \Pr\left\{\frac{Y(t)}{R(t)} \le x\right\} = \frac{1}{\theta^{\varkappa}\Gamma(\varkappa)} \int_0^x x^{\varkappa-1} e^{-x/\theta}, \quad x \ge 0.$$

The limiting distribution of the theorem corresponds to that obtained by Sevastyanov (1957).

Theorem 9. Assume $r(t) \sim t^{\delta} L_R(t), \delta > 0$. Then

$$\lim_{t\to\infty} \mathbf{E}\left[e^{-\lambda Y(t)/\mathbf{E}[Y(t)]}\right] = e^{-\lambda}, \quad \lambda > 0.$$

Corollary. Under the conditions of the theorem $Y(t)/\mathbf{E}[Y(t)] \to 1$ in probability, which can be interpreted as a LLN.

Critical case: CLT is valid

Theorem 10. Assume $r(t) \sim t^{\delta} L_R(t), \delta > 0$ (the conditions of the previous limit theorem). Then CLT

$$X(t) = (Y(t) - \mathbf{E}[Y(t)])/\sqrt{\mathrm{Var}[Y(t)]} \xrightarrow{D} N(0,1), t \to \infty.$$

This can be stated also as

$$\frac{Y(t)}{r(t)t} \sim AN\left(\frac{m_I \mu}{M_a(\delta+1)}, \frac{b(\delta+1)}{M_a(\delta+2)r(t)}\right),$$

which is useful for approximate estimation and statistical inference.

Supercritical case $a > 1 (\alpha > 0)$

Theorem 11. Let $\hat{r}(\alpha) = \lim_{t\to\infty} \int_0^t r(u)e^{-\alpha u}du < \infty$. Then $A(t) \sim m_l C\hat{r}(\alpha)e^{\alpha t}$ as $t \to \infty$ and

$$\zeta(t) = Y(t)/A(t) \xrightarrow{L_2} \zeta \,,$$

ζ is a r.v. with $E\zeta = 1$,

$$\mathrm{Var}(\zeta) = \frac{\hat{r}(2\alpha)\left[m_l W + (m_l + b_l)C^2\right]}{\left[C m_l \hat{r}(\alpha)\right]^2} < \infty,$$

where

$$C = \frac{\int_0^\infty e^{-\alpha t}(1 - G(t))dt}{\int_0^\infty x e^{-\alpha x} a(x) dG(x)} < \infty,$$

$$W = C^2 \frac{\int_0^\infty (b(x) + a(x))e^{-2\alpha x} dG(x) - 1}{1 - \int_0^\infty a(x)e^{-2\alpha x} dG(x)} > 0.$$

Remark. If $r(t) = O(e^{\rho t})$ for some constant $\rho < \alpha$ then $\hat{r}(\alpha) < \infty$.

Theorem 12. Assume $r(t) \sim re^{\rho t}$ with $\rho \geq \alpha$. Then, as $t \to \infty$,
(A) LLN:

$$\zeta(t) = Y(t)/A(t) \xrightarrow{L_2} 1 \ \text{ and } \ \zeta(t) \xrightarrow{a.s.} 1,$$

where $A(t) \sim e^{\rho t} m_I r \int_0^\infty e^{-\rho u} M(u) du$, $\rho > \alpha$,
and $A(t) \sim t e^{\alpha t} m_I r C$, $\rho = \alpha$.
(B) CLT:
(i) If $\alpha \leq \rho \leq 2\alpha$, then $X(t) = [Y(t) - A(t)]/\sqrt{V(t)} \xrightarrow{d} N(0,1)$;
(ii) If $\rho > 2\alpha$, then $X(t) \xrightarrow{d} N(0, \sigma^2)$, where

$$\sigma^2 = 1 - \frac{m_I r \int_0^\infty e^{-\rho u} M(u) du}{r \int_0^\infty e^{-\rho u} [m_I W(u) + (m_I + b_I) M^2(u)] du} < \infty.$$

Concluding remarks.

Concluding remarks. Sevastyanov age-dependent branching processes with non-homogeneous in time Poisson immigration were considered. Limiting behaviour was investigated in subcritical, critical and supercritical cases when the immigration intensity was a regularly varying function or with an exponential growth (positive or negative). The results generalize those obtained for the Markov processes with homogeneous Poisson immigration (Sevastyanov, 1957) and discover also new effects due to inhomogeneity. For example, LLN and CLT were proved in subcritical, critical and supercritical cases.

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