Revisiting limit results for controlled BP

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When you come to a fork in the road, take it.
Yogi Berra

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Controlled Branching Processes (CBP) - discrete stochastic population models

The two qualifiers, discrete and stochastic, simultaneously provide richness and technical challenges in terms of measurements that can be made.

Although the models are relative simple and analytically tractable, they provide understanding behind changing the number of individuals in a population and a guide for developing of more complex models.

The emphasis will be on models' general properties rather than the applicability to any particular real-world system.

φ-Branching Proc., Sevastyanov and Zubkov (1974), TPA

Let I be an (finite or infinite) index set. Define $Z_0 = z_0 > 0$,

$$Z_{n+1} = \sum_{i \in I} \sum_{j=1}^{\varphi_i(Z_n)} \xi_j(n, i), \qquad n \geq 0, \tag{1}$$

where I is an index set and for $i \in I$

$\xi_i = \{\xi_j(n, i)\}$

- i.i.d., ≥ 0, integer-valued r.v.'s, (ind. for different i's).

$\{\varphi_i(n)\} \geq 0$, integer-valued functions.

Population Development in Two Phases

Reproductive Phase: the individuals produce offspring;
Control Phase: the number of potential progenitors is determined.

The individual reproduction law (offspring distribution) is not affected by the control and remains independent of the population size.

$$Z_{n+1} = \sum_{i \in I} \sum_{j=1}^{\varphi_i(Z_n)} \xi_j(n,i), \qquad n \geq 0,$$

A very large class of stochastic processes including

$I = \{1,2\}$, $\varphi_1(n) = n$, and $\varphi_2(n) \equiv 1$
- process with immigration;

$I = \{1,2\}$, $\varphi_1(n) = n$, and $\varphi_2(n) = \max\{1-n, 0\}$
- process with immigration at zero only.

φ-BPs with random φ, N. Yanev (1975), TPA

Define $Z_0 = z_0 > 0$,

$$Z_{n+1} = \sum_{i \in I} \sum_{j=1}^{\varphi_{i,n}(Z_n)} \xi_j(n,i), \qquad n \geq 0, \tag{2}$$

where the random variables $\varphi_i = \{\varphi_{i,n}(k)\}$ are ≥ 0 integer-valued

independent from ξ_i and such that for $i \in I$

$$P(\varphi_{i,n}(k) = j) = p_k(j), \qquad k = 0, 1, \ldots$$

Conditions for Extinction I

$P(Z_n \to 0) + P(Z_n \to \infty) = 1$ the extinction-explosion duality holds.

Let as $k \to \infty$

$$\varphi_n(k) = \alpha_n k \; (1 + o(1)) \quad \text{a.s.},$$

where $\{\alpha_n\}$ are i.i.d. (as α) and independent from the reproduction.

Then (N. Yanev(1975), TPA)

(i) If $E[\log(\alpha E\xi)] < 0$, then $P(Z_n \to 0) = 1$.

(ii) If $E[\log(\alpha E\xi)] > 0$, then $P(Z_n \to 0) < 1$.

T. Bruss (1980), JAP showed that the "independence of reproduction" assumption for $\{\alpha_n\}$ above can be removed.

Conditions for Extinction II

Mean Growth Rate (introduced by T. Bruss (1984), JAP)

$$\tau(k) := \frac{1}{k} E[Z_{n+1} \mid Z_n = k] = \frac{1}{k} E[\varphi_n(k)] E[\xi]$$

For BGW process $\tau(k) = E[\xi]$ -offspring mean.

Denote $q_N := P(Z_n \to 0 \mid Z_0 = N)$ - extinction probability.

Theorem (Gonzalez, Molina, del Puerto (2002), JAP)

(i) If $\limsup_{k\to\infty} \tau(k) < 1$, then $q_N = 1$ for all $N \geq 1$.

(ii) If $\liminf_{k\to\infty} \tau(k) > 1$, then $q_N < 1$ for all $N \geq N_0$.

Classification in terms of the mean growth rate

(Gonzalez, Molina, del Puerto (2005), JAP)

(i) Subcritical - $\limsup_{k\to\infty} \tau(k) < 1$.

(ii) Critical - $\liminf_{k\to\infty} \tau(k) \leq 1 \leq \limsup_{k\to\infty} \tau(k)$.

(iii) Supercritical - $\liminf_{k\to\infty} \tau(k) > 1$.

Unlike the GWP, if the number of ancestors in the supercritical CBP is not sufficiently large, then the extinction probability might be one. This resembles the situation with the two-sex branching processes.

Critical Case

Following Gonzalez, Molina, del Puerto (2005), JAP we write

$$Z_{n+1} = Z_n + g(Z_n) + \alpha_{n+1}, \qquad n = 0, 1, \dots,$$

where $g(Z_n) := E[Z_{n+1}|Z_n] - Z_n$ and

$\alpha_{n+1} := Z_{n+1} - E[Z_{n+1}|Z_n]$ is a martingale difference.

Such stochastic difference eqns were studied by G. Kersting (1992), SPA.

Critical Case I

Case I. Assume

(i) $\tau(k) = 1 + ck^{-1}, \qquad c > 0,\ k = 1, 2, \dots;$

(ii) $E[\alpha_{n+1}^2 | Z_n = k] = ak + O(1), \qquad a > 0;$

(iii) Smoothness assumption on the p.g.f. of φ.

If $2ca^{-1} \leq 1$, then (Gonzalez, Molina, del Puerto (2005), JAP)

$$\lim_{n\to\infty} P\left(\frac{Z_n}{n} \leq z | Z_n > 0\right) = E_{a/2}(z) \quad \text{exponential c.d.f.}$$

Note. The limit over non-extinction trajectories is exponential as in BGW process, even thought the decay rate $P(Z_n > 0) \sim kn^{-(1-2c/a)}$ is different.

Critical Case II

Case II. Assume

$$\text{(i) } \tau(k) = 1 + ck^{-(1-\alpha)} + o\left(k^{-(1-\alpha)}\right), \quad c > 0, 0 < \alpha < 1;$$

$$\text{(ii) } E[\alpha_{n+1}^2 | Z_n = k] = ak^{\beta} + o\left(k^{\beta}\right), \quad \beta \leq \alpha + 1, \; a > 0.$$

Then Z_n, appropriately normalized and conditioned on non-extinction, converges in distribution to either gamma or normal limits depending on the values of α and β.

Branching Processes with Migration

N. Yanev and K. Mitov (1980), C.R. Acad. Bulg. Sci.

$$Y_{n+1} = \sum_{k=1}^{Y_n} \xi_{k,n} + M_n^+ 1_{\{Y_n>0\}} + M_n^0 1_{\{Y_n=0\}}, \; n = 0, 1, \ldots, \; Y_0 \geq 0;$$

where the "migration" is given for $p + q + r = 1$ by

$$M_n^+ = \begin{cases} -e_n & \text{probab.} \quad p, \quad e_n \quad \text{individuals emigrate} \\ 0 & \text{probab.} \quad q, \\ i_n & \text{probab.} \quad r, \quad i_n \quad \text{individuals immigrate.} \end{cases}$$

and

$$M_n^0 = \begin{cases} i_n^0 & \text{probab.} \quad r, \qquad i_n^0 \quad \text{individuals immigrate at 0} \\ 0 & \text{probab.} \quad 1-r, \qquad \text{no migration.} \end{cases}$$

The emigration is regarded as "reversed" (negative) immigration.

Branching Processes with Migration

The particular choice of control functions φ allows for a detailed analysis and interesting new findings.

BPs with migration include previously studied models with different regimes of immigration and emigration.

Key Parameter: mean migration outside 0 over half offspring variance:

$$\theta = \frac{EM_n^+}{(Var\xi)/2}.$$

$$\{Y_n\} = \begin{cases} \text{recurrent} & \theta > 1 \\ \text{null-recurrent} & 0 \le \theta \le 1 \\ \text{positive-recurrent} & \theta < 0. \end{cases}$$

Critical BP with Migration

Under moment assumptions, G. Yanev and N. Yanev (1996), LNS 114

(A) If $\theta > 0$, then $\frac{Y_n}{Var(\xi)n/2} \to \Gamma(\theta, 1)$
- like in BP with immigration only.

If the rate of immigration is not too high, then it serves as a device for maintaining the population.

(B) If $\theta = 0$, then $\frac{\log Y_n}{\log n} \to U(0, 1)$
- like in BP with immigration at 0 only.

(C) If $\theta < 0$, then there is a limiting-stationary distribution, i.e., $Y_n \to Y_\infty$ - new limiting result.

The predominant (on average) emigration leads to a proper (non-quasi) limiting-stationary distribution in the critical case.

Branching Processes with Non-Homogeneous Migration

N. Yanev and K. Mitov (1985), AP

$$Y_{n+1} = \sum_{k=1}^{Y_n} \xi_{k,n} + M_n^{+} 1_{\{Y_n>0\}} + M_n^{0} 1_{\{Y_n=0\}},\ n = 0, 1, \ldots,\ Y_0 \geq 0;$$

where the "migration" is given for $p_n + q_n + r_n = 1$ by

$$M_n^{+} = \begin{cases} -e_n & \text{probab.} \quad p_n \\ 0 & \text{probab.} \quad q_n \\ i_n & \text{probab.} \quad r_n. \end{cases}$$

and

$$M_n^{0} = \begin{cases} i_n^0 & \text{probab.} \quad r_n \\ 0 & \text{probab.} \quad 1 - r_n. \end{cases}$$

Decreasing to 0 and Balanced Migration: $p_n \sim r_n \to 0$

Key Condition: The series $\sum p_k < \infty$ (or $\sum r_k < \infty$).

Theorem 1 Let $p_n \sim r_n \to 0$. If $\sum p_k < \infty$, then

$$\lim_{n\to\infty} P\left(\frac{Y_n}{Var(\xi)n/2} \leq x | Y_n > 0\right) = 1 - e^{-x}, \quad x \geq 0.$$

The convergence of the series $\sum p_k$ and $\sum r_k$ makes the migration disappear without a trace so fast that the limiting result is the same as in BGW process.

Critical BP with Non-Homogeneous Migration II

<u>Theorem 2</u> Let $p_n \sim r_n \to 0$. Assume $\sum p_k = \infty$

and $p_n \sim l_n n^{-1}$, l_n is a s.v.f. at ∞.

If $\lim_{n\to\infty} \frac{l_n \log n}{\sum_{k=1}^{\infty} p_k} =: C < \infty$, then

$$\lim_{n\to\infty} P\left(\frac{\log Y_n}{\log n} \le x | Y_n > 0\right) = \frac{C}{1+C}x, \quad 0 < x < 1 \qquad \text{and}$$

$$\lim_{n\to\infty} P\left(\frac{Y_n}{Var(\xi)n/2} \le x | Y_n > 0\right) = \frac{C}{1+C} + \frac{1}{1+C}\left(1 - e^{-x}\right) \quad x > 0.$$

Critical BP with Non-Homogeneous Migration III

The non-degenerate trajectories of $\{Y_n\}$ are of two types:

$$(i)\ Y_n \sim n^{\eta_1},\ \text{where } \eta_1 \in U(0,1) \text{ with probab. } \frac{C}{1+C};$$

$$(ii)\ Y_n \sim \eta_2 n,\ \text{where } \eta_2 \in Exp\left(\frac{1}{Var\xi/2}\right) \text{ with probab. } \frac{1}{1+C}.$$

This resembles the situation in BP with VE, D'Souza (1994), AAP, and BP with decreasing immigration, Badalbaev and Rahimov (1978), TPA.

Open Problem Find additional conditions, which imply one or another type of non-degenerate trajectories.

Theorem 3 Let $p_n \sim r_n \to 0$. Assume $\sum p_k = \infty$ such that $p_k \sim l_k k^{-v}$ for $0 < v < 1$, then

$$\lim_{n\to\infty} P\left(\frac{\log Y_n}{\log n} \le x | Y_n > 0\right) = x, \quad x \in (0,1).$$

The migration approaches 0 in such a rate that the extinction is certain and the asymptotic behavior on the non-extinction trajectories is as that in Foster-Pakes process with immigration at 0 only.

Alternating Regenerative Process

Regenerative process - from a random time on, is equivalent to what it was at the beginning.

Regenerative processes can be seen as comprising of i.i.d. cycles.

Consider the vector (W, R) with ≥ 0 and independent coordinates and its i.i.d. copies (W_j, R_j) for $j = 1, 2, \dots$

W and R are the working and repairing time periods, respectively, of an operating system.

Denote for $j = 1, 2, \dots$

$$\underset{j\text{th complete cycle}}{T_j} \quad = \quad \underset{j\text{th "repairing" time}}{R_j} \quad + \quad \underset{j\text{th "working" time}}{W_j}$$

Alternating Regenerative Process

For $S_n := \sum_{j=1}^{n} T_j$, define $N(t) := \max\{n \geq 0 : \; S_n \leq t\}$ renewal process

and $\sigma(t) := t - S_{N(t)} - R_{N(t)+1}$.

Associate with each W_j a cycle process $\{z_j(t) : 0 \leq t \leq W_j\}$.

Alternating Regenerative Process (ARP)

$$Z(t) = \begin{cases} z_{N(t)+1}(\sigma(t)) & \text{when } \sigma(t) \geq 0 \quad \text{(machine is up for } \sigma(t) \text{ time)} \\ 0 & \text{when } \sigma(t) < 0 \quad \text{(machine is down).} \end{cases}$$

Migration and Regeneration: Definition

Branching Regenerative Process with Migration

$$Z(t) = \begin{cases} Y^0_{N(t)+1}(\sigma(t)) & \text{when } \sigma(t) \geq 0 \\ 0 & \text{when } \sigma(t) < 0, \end{cases}$$

where $\{Y^0_j(t)\}$ are with migration stopped at 0.

By definition $\sigma(t) := t - S_{N(t)} - R_{N(t)+1}$. That is

$$\begin{array}{ccccccc} S_{N(t)} & \color{red}{t} & & S_{N(t)} + R_{N(t)+1} & & \color{red}{t} & S_{N(t)+1} \\ & & \uparrow & & \uparrow & & \\ & & \color{red}{\sigma(t) < 0} & & \color{red}{\sigma(t) > 0} & & \end{array}$$

Note that R_j are not necessarily geometrically distributed.

Migration and Regeneration: Interpretation

The queueing systems are good examples for regenerative processes.

Consider a queueing model with Poisson arrivals. The service periods are composed of a busy part (non-empty queue) W_j and an idle part (empty queue) R_j. The customers arriving during the service time of a customer are her "offspring". The "immigrants" (probably from another customer pool) will be served in the end of the entire "generation". Alternatively, some "emigrants" may give up and leave the queue.

Migration and Regeneration: Limit Theorems I

Let $\{Z\}$ be critical and $0 < \theta = \frac{EM_t^+}{Var\xi/2} < 1/2$.

Assume $E[R] < \infty$ or $P(R > t) \sim L(t)t^{-\alpha}$ for $\alpha \in (1/2, 1]$.

(G. Yanev, K. Mitov, N. Yanev (2006), J. Appl. Statist. Sci.)

(i) If there is a balance between the working and repairing time, i.e.,

$$0 \leq c := \lim_{t\to\infty} \frac{P(R > t)}{P(W > t)} < \infty,$$

then

$$\begin{aligned} P(\frac{Z_t}{Var(\xi)t/2} \leq x) &= \frac{c}{c+1} \\ &- \frac{1}{(c+1)B(\theta, 1-\theta)} \int_0^1 y^{\theta-1}(1-y)^{-\theta}\left(1 - e^{-x/y}\right) dy. \end{aligned}$$

Migration and Regeneration: Limit Theorems II

(ii) If the repairing time dominates over the working time, i.e.,

$$\lim_{t\to\infty} \frac{P(R > t)}{P(W > t)} = \infty,$$

then

$$P(\frac{Z_t}{Var(\xi)t/2} \leq x) = \frac{1}{B(\theta, \alpha)} \int_0^1 y^{\theta-1}(1-y)^{\alpha-1} \left(1 - e^{-x/y}\right) dy.$$

The distribution of the limit is a mixture of beta and exponential

distributions with mean of $\theta/(\theta + \alpha)$.

BP with Reproduction-Dependent Immigration

Rahimov (1992), TPA

The evolution of the processes initiated by immigrants depends on the "random environment" at the time of immigration.

Let $\eta_{k,n}$ - number of immigrants at time k observed at a later time n.

$\mu_{k,i}(n)$ - the BGW process initiated by the ith immigrant arrived at time k.

Define a BR with Reproduction-Dependent Immigration by:

$$Z_n = \sum_{k=1}^{n} \sum_{i=1}^{\eta_{k,n}} \mu_{k,i}(n-k).$$

The process $\mu_{k,i}(n)$ is called (k,i) process.

BP with Reproduction-Dependent Immigration

Let $w_x^{k,i}$ - number of offspring of individual x from the (k,i) process.

Define

$$\mathcal{F}_{k,i}(n) := \sigma\{w_x^{k,i}, x \in I_{n-k}\}$$

-σ-algebra generated by the evolution of the (k,i) process up to time n.

Assumption on the Immigration For any integer $j \geq 0$ and $0 \leq k \leq n$

$$\{\eta_{k,n} \leq j\} \in F_{k,j}(n) = \prod_{l=1}^{k-1} \prod_{i=1}^{\eta_{l,n}} \mathcal{F}_{l,i}(n) \times \prod_{i=1}^{j} \mathcal{F}_{k,i}(n) \times \mathcal{F}_0,$$

i.e., $\eta_{k,n}$ is a double stopping time with respect to $F_{k,j}$.

BP with Reproduction-Dependent Immigration

Example 1 Let $\xi_k \geq 0$, $k = 0, 1, \ldots$ be independent and integer r.v.s.

Define the immigration by $\eta_0 = \xi_0$ a.s. and for a given set B

$$\eta_k = \begin{cases} \xi_k & \text{when } Z_k^* \in B \\ 0 & \text{when } Z_k^* \notin B, \end{cases} \qquad \text{where} \quad Z_k^* = \sum_{i=0}^{k-1} \sum_{j=1}^{\eta_i} \mu_{i,j}(k-i), \quad k \geq 1.$$

In particular, if $B = \{0\}$, then - BP with immigration at zero only.

BP with Reproduction-Dependent Immigration

Example 2 Let $\xi_k \geq 0$, $k \geq 0$ be integer r.v.s and r_n be positive numbers.

Define the immigration for $0 \leq k \leq n$ by

$$\eta_{k,n} := \max\left\{ j : \sum_{i=1}^{j} \mu_{k,i}(n-k) \leq r_n \xi_k \right\}.$$

We have a process such that approximately the same number of immigrants, who joined the population together, are alive at any time n.

Controlled BP with Continuous State Space

Adke and Gadag (1995), LNS 99, Rahimov (2007), Stoc. Anal. Appl.

Situations when a nonnegative variable (e.g., volume or weight) associated with the individuals is measured.

$$X_{n+1} = \sum_{i=1}^{N_{n+1}(X_n)} W_i(n+1) + U(n+1),$$

where $\{W_i(n)\}$ and $\{U(n)\}$ are ≥ 0 but non-integer-valued and $\{N_n(x)\}$ are counting processes with independent stationary increments. The three sets are mutually independent.

Recent monograph:

S. Aliev, Y.I. Yeleyko, and I.B. Bazylevych (2010). Limit Theorems and Transitional Phenomena in the Theory of BPs, VNTL Publishers, Ukraine.

More Controlled Branching Processes

(i) del Puerto and N. Yanev (2004), J. Appl. Statist. Sci. - Multitype CBP.

(ii) P. Mayster (2005), JAP introduced the Alternating BP - controlling a BP by means of another BP.

(iii) Gonzalez, Minuesa, Mota, del Puerto, and Ramos (2015), Lithuanian Math. J. - CBP in varying environment.

(iv) T. Bruss and M. Duerlinckx (2015), AAP - Resource-Dependent BP.

Concluding Remarks

This survey is by no means exhaustive. Not included here are some classes CBPs such as: branching processes with barriers (see Zubkov (1972), Bruss (1978), Schuh (1976), Sevastyanov (1995)), and CBPs with random environments.

Closed relations were established between CBP and other classes, e.g., two-sex processes and population-size dependent processes.

There is no doubt, that the CBP have a great potential as modeling tools.

In my opinion, they deserve more attention from the branching processes' community.

Concluding Remarks

We paid special attention to the processes with migration, which have been a subject of systematical research investigations by the Bulgarian School in branching processes under the direction of its founder Professor Nikolay Yanev a.k.a. the Captain.