Subcritical branching processes in random environment

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and under the influence of

V.Afanasyev, Ch.Boeinghoff, J.Geiger and G.Kersting

Badajoz, April 2015

Galton-Watson processes in random environment

- Offspring generating functions $f_n(s) := \mathbf{E}s^{\xi^{(n)}}$ in generations $n = 0, 1, \ldots$ are RANDOM and I.I.D.

  $\Rightarrow f'_n(1) := \mathbf{E}\xi^{(n)}$ are I.I.D. RANDOM variables;

- $$Z_{n+1} = \sum_{j=1}^{Z_n} \xi_j^{(n)} \quad or \quad \mathbf{E}\left[s^{Z_{n+1}} \,\middle|\, Z_n; f_0, f_1, \ldots\right] = \left(f_n(s)\right)^{Z_n}.$$

- $\xi_j^{(n)} \stackrel{d}{=} \xi^{(n)}$ are i.i.d. **given** $f_0, f_1, \ldots$

Associated random walk

- $X_i := \log f'_{i-1}(1)$
- $S_0 = 0, \qquad S_n = \log f'_0(1) + \log f'_1(1) + \cdots + \log f'_{n-1}(1), \quad n \geq 1.$

$$\mathbf{E}\Big[Z_n \Big| f_0, f_1, \ldots\Big] =: \mathbf{E}_f\Big[Z_n\Big] = \prod_{j=0}^{n-1} f'_j(1) = e^{S_n}$$

Associated random walk

- $X_i := \log f'_{i-1}(1)$
- $S_0 = 0, \qquad S_n = \log f'_0(1) + \log f'_1(1) + \cdots + \log f'_{n-1}(1), \quad n \geq 1.$

$$\mathbf{E}_f \Big[Z_n \Big] = e^{S_n}$$

Classification (Afanasyev, Geiger, Kersting, Vatutin (2005), Ann. of Probab.): A BPRE is called

- supercritical if $\lim_{n\to\infty} S_n = +\infty$ with probability 1;
- subcritical if $\lim_{n\to\infty} S_n = -\infty$ with probability 1;
- critical if $\limsup_{n\to\infty} S_n = +\infty$ and $\liminf_{n\to\infty} S_n = -\infty$ (both with probability 1);

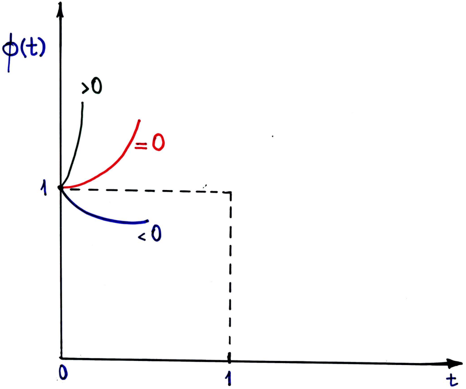
- $X_i := \log f'_{i-1}(1)$
- $S_0 = 0, \qquad S_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1.$

We may express the classification in terms of the moment generating function:

$$\phi(t) = \mathbf{E}e^{tX} = \mathbf{E}e^{t \log f'(1)}.$$

Classification: A BPRE is called

- supercritical if $\phi'(0) = \mathbf{E}X = \mathbf{E}\log f'(1) > 0$,
- critical if $\phi'(0) = \mathbf{E}X = \mathbf{E}\log f'(1) = 0$ or $\phi'(0)$ does not exist,
- subcritical if $\phi'(0) = \mathbf{E}X = \mathbf{E}\log f'(1) < 0$.



$$\phi(t) = Ee^{tX}$$

Quenched approach:

The study the behavior of characteristics of a BPRE for **typical realizations** of the environment $f_0, f_1, ...$

For instance, if

$$T = \min\{n : Z_n = 0\}$$

then

$$\mathbf{P}(T > n | f_0, f_1, ...) = \mathbf{P}_f(T > n) = \mathbf{P}_f(Z_n > 0)$$

is a random variable.

Annealed approach:

The study the behavior of characteristics of a BPRE performing **averaging over possible scenarios** $f_0, f_1, ...$ on the space of realizations of the environment:

$$\mathbf{P}(Z_n > 0) = \mathbf{E}[\mathbf{P}_f(Z_n > 0)]$$

is a number.

ANNEALED APPROACH

SUBCRITICAL PROCESSES

$$\mathbf{E}X = \mathbf{E}\log f'(1) < 0$$

The problems to be considered **under the annealed approach**:

- Survival probabilities $\mathbf{P}(Z_n > 0)$ as $n \to \infty$;

- The environments providing survival $\mathbf{P}(S_m \in dx | Z_n > 0)$ as $n \to \infty$;

- Conditional limit theorems $\mathbf{P}(Z_{nt} \in dx | Z_n > 0)$ as $n \to \infty$.

Subcritical processes: **five** different sub-cases:

- strongly subcritical, if

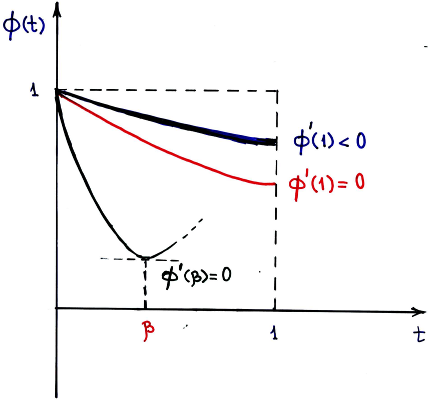
$$\phi'(1) = \mathbf{E}\left[Xe^{X}\right] < 0,$$

- intermediately subcritical, if

$$\phi'(1) = \mathbf{E}\left[Xe^{X}\right] = 0,$$

- weakly subcritical, if there exists $0 < \beta < 1$ such that

$$\phi'(\beta) = \mathbf{E}\left[Xe^{\beta X}\right] = 0.$$



ϕ(t)
1
ϕ'(1) < 0
ϕ'(1) = 0
ϕ'(β) = 0
β
1
t

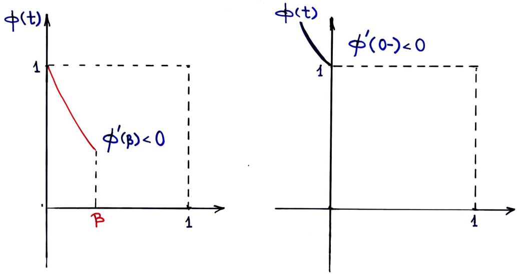
Subcritical processes ($\mathbf{E}X = \mathbf{E}\log f'(1) < 0$): **five** different sub-cases:

- Non-Cramer subcritical, if

$$\beta = \sup\{t \geq 0 : \mathbf{E}\left[e^{tX}\right] < \infty\} \in [0, 1)$$

  and

$$\phi'(\beta) = \mathbf{E}\left[Xe^{\beta X}\right] < 0.$$

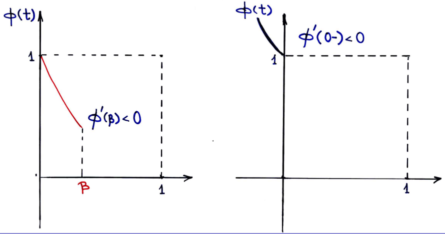


Two examples: the first $\beta \in (0,1)$:

$$p_X(x) = \begin{cases} \frac{c}{(|x|+1)^4} & \text{if} \quad x < 0; \\ \frac{c}{(|x|+1)^4} e^{-\beta x} & \text{if} \quad x \geq 0. \end{cases}$$

The second $\beta = 0$:

$$p_X(x) = \begin{cases} \frac{c}{(|x|+1)^4} & \text{if} \quad x < 0; \\ \frac{c}{(|x|+1)^5} & \text{if} \quad x \geq 0. \end{cases}$$



Change of measure for ordinary RW meeting the **Cramer** condition:

$\phi(t) = \mathbf{E}e^{tX}$ and $\phi'(\beta) = \mathbf{E}Xe^{\beta X} = 0$

Change of measure

$Y_n = \varphi(S_1, S_2, ..., S_n)$

$\Rightarrow$ New measure: $\hat{\mathbf{P}}_\beta, \beta > 0$ or $\hat{\mathbf{E}}_\beta$

$$\hat{\mathbf{E}}_\beta Y_n = \frac{\mathbf{E}Y_n e^{\beta S_n}}{\mathbf{E}e^{\beta S_n}} = \frac{\mathbf{E}Y_n e^{\beta S_n}}{\phi^n(\beta)}$$

with

$$\hat{\mathbf{E}}_\beta X = \frac{\mathbf{E}Xe^{\beta X}}{\mathbf{E}e^{\beta X}} = \frac{\phi'(\beta)}{\phi(\beta)} = 0$$

The change of measure for branching processes in RE

$Y_n = \varphi(f_1, \ldots, f_{n-1}; Z_0, \ldots, Z_n)$

We introduce a family $\hat{\mathbf{P}}_\beta, \beta > 0$ of probability measures, given by

$$\hat{\mathbf{E}}_\beta[Y_n] = \frac{\mathbf{E}[Y_n e^{\beta S_n}]}{\mathbf{E}[e^{\beta S_n}]} = \gamma^{-n}\mathbf{E}[Y_n e^{\beta S_n}]$$

with

$$\gamma = \gamma_\beta = \mathbf{E}[e^{\beta X}]$$

or

$$\mathbf{E}[Y_n] = \gamma^n \hat{\mathbf{E}}_\beta[Y_n e^{-\beta S_n}].$$

It is necessary to select β in such a way for

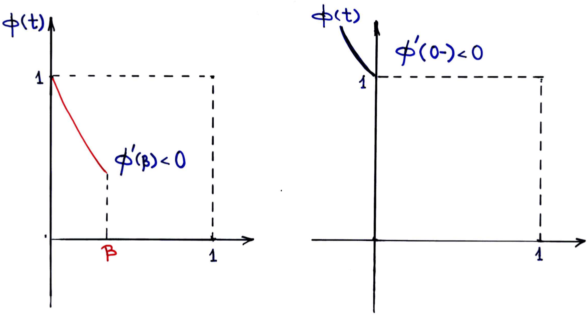
$$\hat{\mathbf{E}}_\beta[X] = \frac{\mathbf{E}[X e^{\beta X}]}{\mathbf{E}[e^{\beta X}]} = 0$$

- Non-Cramer subcritical, if

$$\beta = \sup\{t \geq 0 : \mathbf{E}\left[e^{tX}\right] < \infty\} \in [0,1)$$

and

$$\phi'(\beta) = \mathbf{E}\left[Xe^{\beta X}\right] < 0.$$



Non-Cramer subcritical

$\beta = \sup\{t \geq 0 : \mathbf{E}\left[e^{tX}\right] < \infty\} \in [0,1)$ and $\phi'(\beta) = \mathbf{E}\left[Xe^{\beta X}\right] < 0.$

The case $\beta = 0:$

$$\phi'(0-) = \mathbf{E}X = \mathbf{E}\log f'(1) = -a < 0$$

and

$$\phi(t) = \mathbf{E}e^{tX} = \infty$$

for all $t > 0.$

Non-Cramer subcritical

(V.+Xinghua Zheng) The case $\beta = 0:$

Hypothesis A.

$$\mathbf{E}X = -a < 0, \qquad \mathbf{P}\left(X > x\right) \sim \frac{l(x)}{x^b}, b > 2.$$

Non-Cramer subcritical

(V.+Xinghua Zheng) The case $\beta = 0:$

Hypothesis A.

$$\mathbf{E}X = -a < 0, \qquad \mathbf{P}\left(X > x\right) \sim \frac{l(x)}{x^b}, b > 2.$$

Let $f'(1) = \mathbf{E}_f[\xi] = e^X$.

Hypothesis B. For **each** $\lambda > 0$ as $y \to \infty$,

$$\mathcal{L}\left(\mathbf{E}_f\left[\exp\left\{-\lambda\frac{Z_1}{y}\right\}\right] | f'(1) > y\right) \xrightarrow{d} \mathcal{L}(\gamma),$$

where γ is a random variable which is less than 1 with a positive probability and is **independent** of λ.

Hypothesis B. If

$$f(s)=\sum_{k=0}^{\infty}p_k(f)s^k$$

then the sequence of random distributions

$$\mathbf{P}_{y,f}(x)=\mathbf{P}_f\left(\frac{Z_1}{y}\le x\right)=\sum_{0\le k\le xy}p_k(f)$$

possesses the property: conditionally on $f'(1)>y$ for any $x>0$

$$\mathbf{P}_f\left(\frac{Z_1}{y}\le x\right)\xrightarrow{d}\gamma<1$$

as $y\to\infty$. Thus, the limiting measure is

$$\delta_0\gamma+(1-\gamma)\,\delta_\infty.$$

$$\mathbf{E}X = -a < 0, \qquad \mathbf{P}\left(X > x\right) \sim \frac{l(x)}{x^b}, b > 2.$$

If Hypothesis A and B are valid then

$$\mathbf{P}\left(Z_n > 0\right) \sim K\mathbf{P}\left(X > na\right) \sim K_1\mathbf{P}\left(\min_{0\le i\le n} S_i \ge 0\right).$$

Let

$$U_n = \inf \{j : X_j > na\}$$

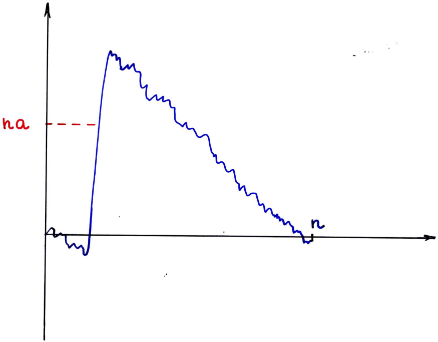
be **the moment of the first big jump.**
Then

$$\lim_{n\to\infty} \mathbf{P}(U_n = j | Z_n > 0) = p_j, \quad \sum_{j=1}^{\infty} p_j = 1.$$

In fact, we have **only one big jump exceeding** na, and

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathbf{P}(S_{U_n} > na - M\sqrt{n} | Z_n > 0) = 1.$$

Non-Cramer subcritcal $\beta = 0$



Non-Cramer subcritcal $\beta = 0$
Moreover, for any $\varepsilon \in (0,1)$

$$\mathcal{L}\left(\frac{Z_{nt}}{Z_{U_n}\exp(S_{nt} - S_{U_n})}, \varepsilon \le t \le 1 \middle| Z_n > 0 \right) \Longrightarrow \mathcal{L}\left(1, \varepsilon \le t \le 1\right),$$

and

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}} \left(\log \frac{Z_{nt}}{Z_{U_n}} + nta \right), \varepsilon \le t \le 1 \middle| Z_n > 0 \right) \Longrightarrow \mathcal{L}(B_t, \varepsilon \le t \le 1),$$

where B_t is a Brownian motion.

Non-Cramer subcritcal $\beta \in (0, 1)$

$\beta = \sup\{t \geq 0 : \mathbf{E}\left[e^{tX}\right] < \infty\} \in [0, 1)$ and $\phi'(\beta) = \mathbf{E}\left[Xe^{\beta X}\right] < 0.$

Hypothesis A1. The distribution of X has density

$$p_X(x) = \frac{l_0(x)}{x^{b+1}} e^{-\beta x},$$

where $l_0(x)$ is a function slowly varying at infinity, $b > 2$, $\beta \in (0, 1)$.

Hypothesis A1. The distribution of X has density

$$p_X(x) = \frac{l_0(x)}{x^{b+1}} e^{-\beta x},$$

where $l_0(x)$ is a function slowly varying at infinity, $b > 2$, $\beta \in (0,1)$.

Hypothesis B1. There exists a random function

$$g(\lambda), \lambda \in [0, \infty), 0 < g(\lambda) < 1$$

for all $\lambda > 0$, and $\lim_{\lambda \to \infty} g(\lambda) = 0$ such that

$$\mathcal{L}\left(\mathbf{E}_f\left[\exp\left\{-\lambda \frac{Z_1}{y}\right\}\right] | f'(1) = y\right) \xrightarrow{d} \mathcal{L}(g(\lambda)).$$

Hypothesis B1. If

$$f(s) = \sum_{k=0}^{\infty} p_k(f) s^k$$

then the sequence of random distributions

$$\mathbf{P}_{y,f}(x) = \mathbf{P}_f\left(\frac{Z_1}{y} \leq x\right) = \sum_{0 \leq k \leq xy} p_k(f)$$

possesses the property: conditionally on $f'(1) = y$ for any $x > 0$

$$\mathbf{P}_f\left(\frac{Z_1}{y} \leq x\right) \xrightarrow{d} \mathbf{P}_\omega\left(\xi \leq x\right)$$

as $y \to \infty$ and the random distribution $\mathbf{P}_\omega\left(\xi \leq x\right)$ is proper and nondegenerate with probability 1.

(V.+ Vincent Bansaye) The case $\beta \in (0,1)$:

Let

$$a = -\frac{\phi'(\beta)}{\phi(\beta)} > 0.$$

If Hypothesis A1 and B1 are valid then

$$\mathbf{P}(Z_n > 0) \sim C_1 \mathbf{P}\left(\min_{0 \le k \le n} S_k \ge 0\right) \sim C\phi^n(\beta)\frac{l_0(n)}{(an)^{b+1}}.$$

Let

$$U_n = \inf\left\{j : X_j > na(1-\varepsilon)\right\}, \quad \varepsilon \in (0,1).$$

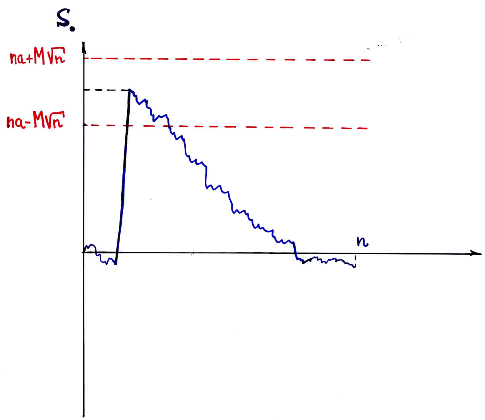
be **the moment of the first big jump.**
Then for **any** $\varepsilon \in (0,1)$

$$\lim_{n\to\infty} \mathbf{P}(U_n = j | Z_n > 0) = p_j, \quad \sum_{j=1}^{\infty} p_j = 1.$$

and

$$\lim_{M\to\infty} \lim_{n\to\infty} \mathbf{P}(S_{U_n} \in [na - M\sqrt{n}, na + M\sqrt{n}] | Z_n > 0) = 1.$$

Non-Cramer subcritcal $\beta > 0$



For any $s \in [0, 1]$

$$\lim_{n \to \infty} \mathbf{E}[s^{Z_n} | Z_n > 0] = F(s) = \sum_{k=1}^{\infty} p_k s^k,$$

with $\sum_{k=1}^{\infty} p_k = 1$.

The change of measure for branching processes in RE

$Y_n = \varphi(f_1, \ldots, f_{n-1}; Z_0, \ldots, Z_n)$

We introduce a family $\hat{\mathbf{P}}_\beta, \beta > 0$ of probability measures, given by

$$\hat{\mathbf{E}}_\beta[Y_n] = \frac{\mathbf{E}[Y_n e^{\beta S_n}]}{\mathbf{E}[e^{\beta S_n}]} = \gamma^{-n}\mathbf{E}[Y_n e^{\beta S_n}]$$

with

$$\gamma = \gamma_\beta = \mathbf{E}[e^{\beta X}]$$

or

$$\mathbf{E}[Y_n] = \gamma^n \hat{\mathbf{E}}_\beta[Y_n e^{-\beta S_n}].$$

It is necessary to select β in such a way for

$$\hat{\mathbf{E}}_\beta[X] = \frac{\mathbf{E}[X e^{\beta X}]}{\mathbf{E}[e^{\beta X}]} = 0$$

$$\mathbf{E}[Y_n] = \gamma^n \hat{\mathbf{E}}_\beta[Y_n e^{-\beta S_n}]$$

Thus, for

$$\gamma = \gamma_\beta = \mathbf{E}[e^{\beta X}]$$

$$\mathbf{P}(Z_n > 0) = \mathbf{E}[\mathbf{P}_f(Z_n > 0)] = \gamma^n \hat{\mathbf{E}}_\beta[\mathbf{P}_f(Z_n > 0) e^{-\beta S_n}].$$

One can show that under mild conditions

$$\mathbf{P}_f(Z_n > 0) \asymp e^{\min_{0 \le k \le n} S_k}.$$

Thus,

$$\mathbf{P}(Z_n > 0) \asymp \gamma^n \hat{\mathbf{E}}_\beta[e^{\min_{0 \le k \le n} S_k - \beta S_n}].$$

Intermediately subcritical **(Afanasyev, Boeinghoff, Kersting+V.)**:

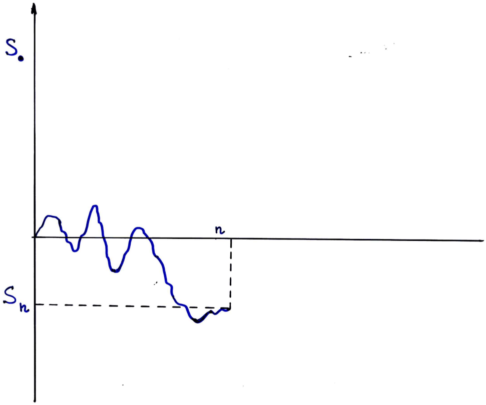
$$\frac{\phi'(1)}{\phi(1)} = \frac{\mathbf{E}\left[Xe^{X}\right]}{\mathbf{E}\left[e^{X}\right]} = \hat{\mathbf{E}}_1[X] = 0,$$

and

$$\frac{\mathbf{E}\left[X^2e^{X}\right]}{\mathbf{E}\left[e^{X}\right]} = \hat{\mathbf{E}}_1[X^2] \in (0, \infty).$$

Then one should take $\beta = 1$, the random walk is recurrent under $\hat{\mathbf{P}}_1$

$$\hat{\mathbf{E}}_1[e^{\min_{0\le k\le n} S_k - S_n}] \asymp \hat{\mathbf{P}}_1(\min_{0\le k\le n} S_k \approx S_n).$$



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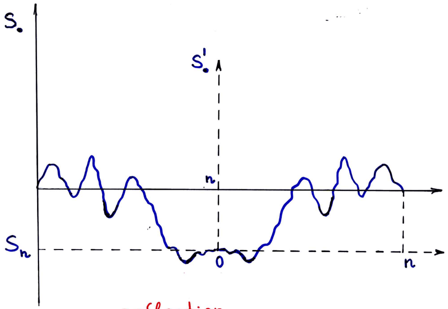


reflection

Intermediately subcritical:

$$\frac{\phi'(1)}{\phi(1)} = \frac{\mathbf{E}\left[Xe^X\right]}{\mathbf{E}\left[e^X\right]} = \hat{\mathbf{E}}_1[X] = 0,$$

and

$$\frac{\mathbf{E}\left[X^2e^X\right]}{\mathbf{E}\left[e^X\right]} = \hat{\mathbf{E}}_1[X^2] \in (0,\infty).$$

Then $\beta = 1$, the random walk is recurrent under $\hat{\mathbf{P}}_1$

$$\hat{\mathbf{P}}_1(\min_{0\le k\le n} S_k \approx S_n) \approx \hat{\mathbf{P}}_1(\min_{0\le k\le n} S'_k \ge -a) \sim \frac{C}{\sqrt{n}}, \quad a>0.$$

Intermediately subcritical:
Hence,

$$\mathbf{P}(Z_n > 0) \asymp \gamma^n \hat{\mathbf{E}}_1[e^{\min_{0\le k\le n} S_k - S_n}] \asymp \gamma^n \hat{\mathbf{P}}_1(\min_{0\le k\le n} S_k \approx S_n) \sim C\frac{\gamma^n}{\sqrt{n}}.$$

Intermediately subcritical:

Trajectories of $\{S_n, n \geq 0\}$ that provide survival:

Let $B^* = (B_t^*, 0 \leq t \leq 1)$ denote a Brownian motion on $[0,1]$ conditioned to attain its minimum at time $t = 1$. Then

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}} S_{nt}, 0 \leq t \leq 1 | Z_n > 0\right) \Rightarrow \mathcal{L}\left(B_t^*, 0 \leq t \leq 1\right).$$

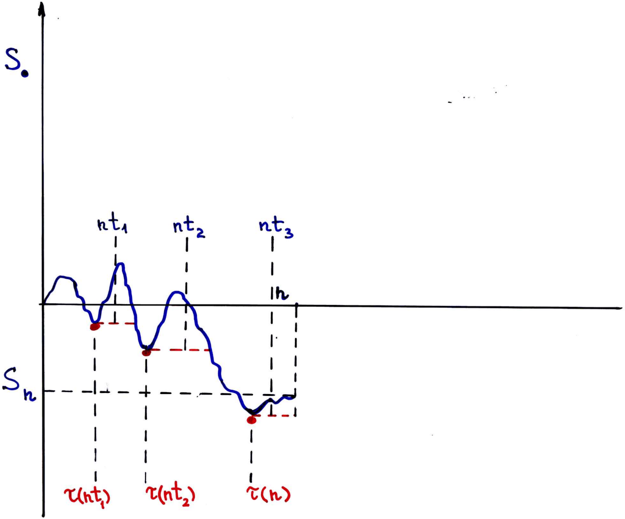
Intermediately subcritical

Let $\tau(m)$ be the moment at which the minimal value of the random walk $\{S_k, 0 \leq k \leq m\}$ is attained.

Assume that for $0 < t_1 < \ldots < t_k < 1$ the random moments

$$\tau(nt_1) < \tau(nt_2) < ... < \tau(nt_k)$$

are increasing.



S.
nt1
nt2
nt3
n
Sn
τ(nt1)
τ(nt2)
τ(n)

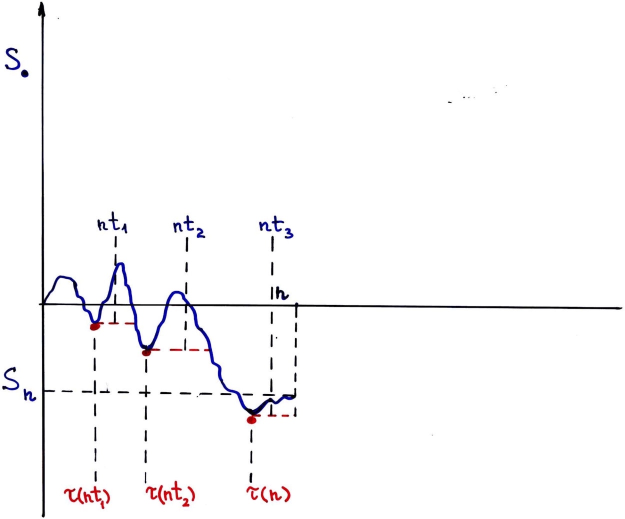
Intermediately subcritical: $\mathbf{E}\left[Xe^{X}\right]=0$
Let $\tau(m)$ be the moment at which the minimal value of the random walk $\{S_k, 0\leq k\leq m\}$ is attained.
Assume that for $0<t_1<\ldots<t_k<1$ the random moments

$$\tau(nt_1)<\tau(nt_2)<...<\tau(nt_k).$$

are increasing. Then

$$\mathcal{L}\left(\frac{Z_{nt_1}}{e^{S_{nt_1}-S_{\tau(nt_1)}}},...,\frac{Z_{nt_k}}{e^{S_{nt_k}-S_{\tau(nt_k)}}}\,|Z_n>0\right)\xrightarrow{d}\mathcal{L}\left(V_1,\ldots,V_k\right)$$

where $V_1,V_2,\ldots$ are i.i.d. copies of a strictly positive random variable V.



$S_\bullet$
nt_1
nt_2
nt_3
n
S_n
$\tau(nt_1)$
$\tau(nt_2)$
$\tau(n)$

Intermediately subcritical:

$$\mathcal{L}\left(\frac{Z_{nt_1}}{e^{S_{nt_1}-S_{\tau(nt_1)}}},...,\frac{Z_{nt_k}}{e^{S_{nt_k}-S_{\tau(nt_k)}}}\,|Z_n>0\right)\xrightarrow{d}\mathcal{L}\left(V_1,\ldots,V_k\right)$$

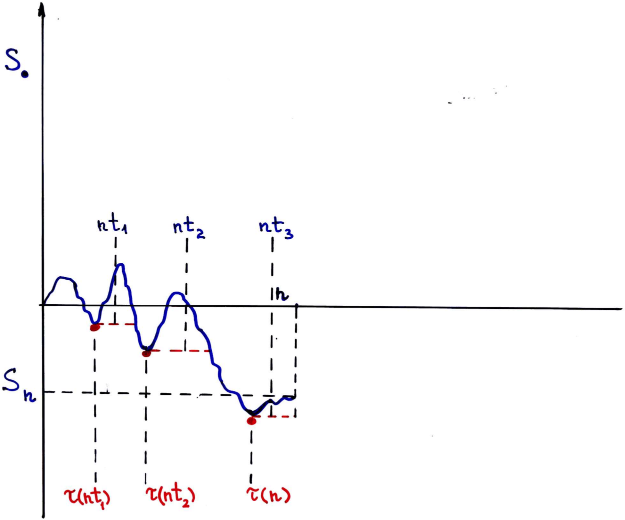
where $V_1, V_2, \ldots$ are i.i.d. copies of a strictly positive random variable V.

In addition,

$$\Big((Z_{\tau(nt_1)},\ldots,Z_{\tau(nt_k)})|Z_n>0\Big)\xrightarrow{d}(W_1,\ldots,W_k)$$

where $W_1, W_2, \ldots$ are i.i.d. copies of some strictly positive **integer-valued** random variable W.

BOTTLENECKS!!



Weakly subcritical **(Afanasyev, Boeinghoff, Kersting+V.)**:

There exists $0 < \beta < 1$ such that

$$\mathbf{E}\left[X e^{\beta X}\right] = 0, \qquad \mathbf{E}\left[X^2 e^{\beta X}\right] \in (0, \infty).$$

Then $\gamma := \mathbf{E}\left[e^{\beta X}\right]$.

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Then $\gamma := \mathbf{E}\left[e^{\beta X}\right]$ and

$$\mathbf{P}\left(Z_n > 0\right) = \gamma^n \hat{\mathbf{E}}_\beta \left[\mathbf{P}_f(Z_n > 0) e^{-\beta S_n}\right] \asymp \gamma^n \hat{\mathbf{E}}_\beta \left[e^{\min(S_0, S_1, \ldots, S_n) - \beta S_n}\right]$$

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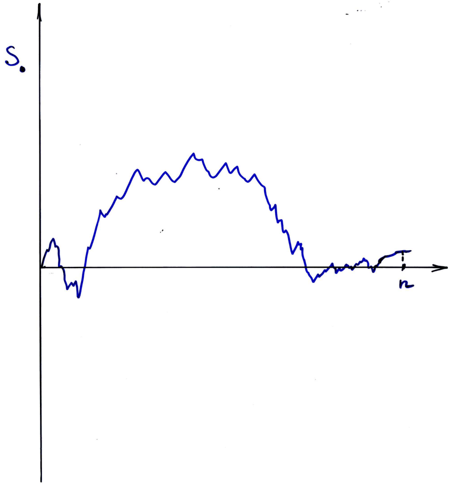
and

$$\hat{\mathbf{E}}_\beta \left[e^{\min(S_0, S_1, \ldots, S_n) - \beta S_n}\right] \asymp \hat{\mathbf{P}}_\beta \left(\min\left(S_0, S_1, ..., S_n\right) \approx 0, S_n \approx 0\right) \asymp n^{-3/2}.$$



$S_\bullet$
n

Therefore,

$$\mathbf{P}\left(Z_n > 0\right) \asymp \gamma^n \hat{\mathbf{E}}_\beta \left[e^{\min(S_0, S_1, \ldots, S_n) - \beta S_n} \right] \sim \gamma^n \frac{C}{n^{3/2}}.$$

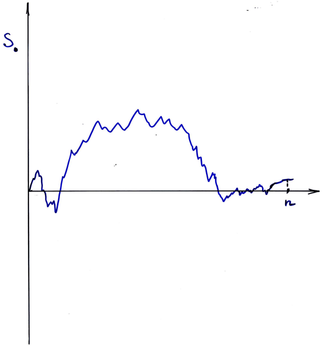
In fact, it follows from **Borovkov**[3] that

$$\mathbf{P}\left(Z_n > 0\right) \sim \gamma^n \frac{C}{n^{3/2}} \sim C_2 \mathbf{P}\left(\min_{0 \le k \le n} S_k \ge 0 \right).$$

Environment providing survival

$$\mathcal{L}\left(\frac{S_{nt}}{\sigma\sqrt{n}}, 0 \le t \le 1 \,|\, Z_n > 0\right) \Rightarrow \mathcal{L}\left(B_t^0, 0 \le t \le 1\right),$$

where $B_t^0, 0 \le t \le 1$ is the Brownian excursion.



Limit theorems

$$\mathcal{L}(Z_n | Z_n > 0) \to \mathcal{L}(\zeta).$$

where ζ is a proper integer-valued r.v.

For each $\varepsilon \in (0, 1/2)$

$$\mathcal{L}\left(\frac{Z_{nt}}{e^{S_{nt}}}, \varepsilon \le t \le 1 - \varepsilon \,|Z_n > 0\right) \Rightarrow \mathcal{L}\left(W_t, \varepsilon \le t \le 1 - \varepsilon\right),$$

where $W_t = W, 0 < t < 1$ a.s. and $\mathbf{P}(0 < W < \infty) = 1$.

Strongly subcritical **(Afanasyev, Geiger, Kersting+V.)**:

$$\frac{\phi'(1)}{\phi(1)} = \mathbf{E}\left[Xe^{X}\right] = \hat{\mathbf{E}}_1[X] < 0.$$

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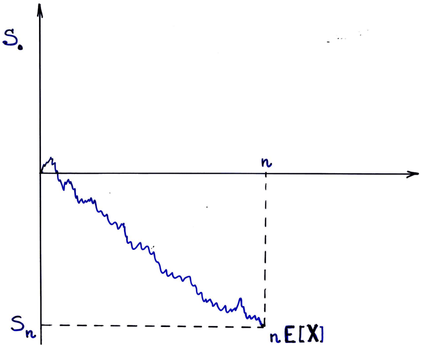
Note that

$$0 > \hat{\mathbf{E}}_1[X] = \frac{\mathbf{E}\left[Xe^X\right]}{\mathbf{E}[e^X]} > \mathbf{E}[X].$$



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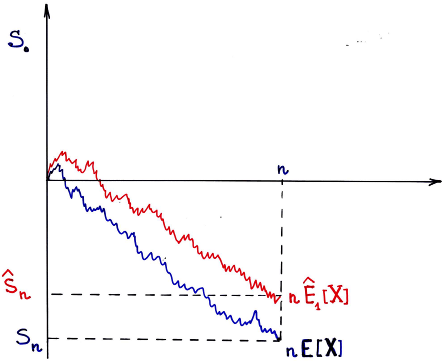

$S_\bullet$
n
$\hat{S}_n$
$n\,\hat{E}_1[X]$
S_n
$n\,E[X]$

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Then $\beta = 1$

$$\hat{\mathbf{E}}_1[e^{\min_{0\le k\le n} S_k - S_n}] \asymp \hat{\mathbf{P}}_1(\min_{0\le k\le n} S_k \approx S_n) \approx C \in (0,\infty).$$

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Hence,

$$\mathbf{P}(Z_n > 0) \sim C\gamma^n.$$

For every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{0\le t\le 1}\left|\frac{1}{n}S_{nt} - t\hat{\mathbf{E}}\left[X\right]\right| > \varepsilon | Z_n > 0\right) = 0.$$

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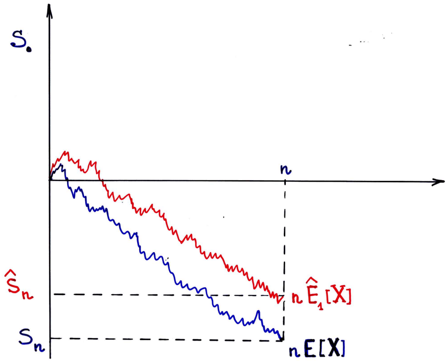
and for $0 < t_1 < \ldots < t_k < 1$

$$\mathbf{P}((Z_{nt_1}, \ldots, Z_{nt_k})|Z_n > 0) \xrightarrow{d} (W_1, \ldots, W_k)$$

where $W_1, W_2, \ldots$ are I.I.D. copies of some strictly positive integer-valued random variable W.



$S_{\cdot}$
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S_n
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THANKS!