Crump-Mode-Jagers Branching Process: A Numerical Approach

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Uses of GBP

Modelling human population

The GBP allows for a woman to have many births during her life at different times. This flexibility allows the model to be used in practice.

Use in demographics

We can use the GBP to calculate the Malthusian parameter of the population and how it evolved in time.

Forecasting

The model gives us the expected future population, i.e. a forecast. In addition because of the continues time of the model this forecast is actually a little more accurate than the usual discrete demographic forecast made on yearly basis.

The Standard General Branching Model

Life length

The life length of a person is modeled as random variable λ_x, where x is an index. (can be thought of as the ID number or n-tuple).

Birth process

A natural candidate for a model of birth process is the **point process**. It is a stochastic process describing the number of children a woman has in every moment of her life. It is denoted by ξ_x (x is the ID).

Demographic Interpretation

$\xi(t)$ - child count of woman aged t. The number of children born to a woman in the age interval $[a, b]$ is denoted by $\xi[a, b]$ and is an integer random variable.

The expected number of births is $\mu[a, b] = \mathbb{E}(\xi[a, b])$. It is the average number of children born to a woman in this age interval. It is natural to assume $\mu(t)$ to be a smooth function. We can assume that children can't give birth so $\xi(t) = 0$ and $\mu(t) = 0$ for $t < 12$.

For a human population is natural to assume a smooth function for distribution of time of death - $L(t)$. So the survivability $S(t) = 1 - L(t)$ is smooth too. We can assume that $S(0) = 1$ и $S(\omega) = 0$, where ω is the greatest age in the life tables.

The General Branching Process

Let σ_x is the date of birth for individual x. We define an indicator $z_t^a(x)$ whether x is alive and younger than $a > 0$ at time $t > 0$.

$$z_t^a(x) = \begin{cases} 1, \text{when } t - a < \sigma_x \leq t < \sigma_x + \lambda_x \\ 0, \text{otherwise} \end{cases}$$

Definition 1

General Branching Process (GBP) is

$$z_t^a = \sum_{x \in I} z_t^a(x),$$

where I is the index set of all n-tuples, for all n.

The General Branching Process

Let us denote $f(s) = \mathbb{E}(s^{\xi(\infty)})$, $|s| \leq 1$, $L(t) = \mathbb{P}(\lambda_x \leq t)$, $\hat{\mu}$ is the Laplace-Stieltjes transformation of $\mu(t) = \mathbb{E}\xi(t)$ and $S(t) = 1 - L(t)$.

Theorem 1

(see Jagers, Branching Processes with Biological Application, Chapter 6, 1975)
If $f(s) < \infty, |s| \leq 1$, then $m_t = \mathbb{E}(z_t) < \infty, \forall t$ and $m_t^a = \mathbb{E}(z_t^a)$ satisfies

$$m_t^a = 1_{[0,a)}(t)\{1 - L(t)\} + \int_0^t m_{t-u}^a \mu(du). \qquad (2.1)$$

Adjusted GBP. Notation

Let ${}_b z_t^a$ be the number of individuals younger than a at time t, that started from a woman aged b at time 0. When $a = \infty$ and $b = 0$ we can skip a and b so z_t denotes the population at time t started from a woman aged 0 at time 0.

Let ${}_b\xi$ is her point process and ${}_b\mu$ is the expectation of the point process

Let ${}_bS$ is her survivability function.

Let $n_b = \mathbb{P}(\xi\,[b, b+1) = 1 | \lambda \geq b)$ be the probability a woman to give birth at age b if she survived to the beginning of this age interval.

We have that ${}_b\mu(t+b) = \mathbb{E}(\xi(t+b)|\lambda > b), t > b$ and ${}_bS(t) = \mathbb{P}(\lambda > b + t | \lambda > b)$.

Previous Results

We can model ${}_b\mu(t)$ from the data:

Proposition 1

(see Trayanov, Pliska Stud. Math. Bulgar. 22, 2013)
For $k \geq 1$ the distribution of ${}_b\xi$ satisfies

$$\begin{aligned}\mathbb{P}({}_b\xi[b+k-1, b+k) = 1) &= 1 - \mathbb{P}({}_b\xi[b+k-1, b+k) = 0) \\ &= {}_bS(b+k-1) \cdot n_{b+k-1}\end{aligned}$$

and the expected number of births in $[b+k-1, b+k)$ of a woman aged b is

$${}_b\mu[b+k-1, b+k) = {}_bS(b+k-1) \cdot n_{b+k-1}$$

Results

Theorem 2

Let $1_{[0,a-b)}(t)$ is an indicator function: $1_{[0,a-b)}(t) = 1$, when $a - b > 0$ and $t \in [0, a-b)$; $1_{[0,a-b)}(t) = 0$ when $a - b < 0$ or $t \geq a - b$.
If $F(s) < \infty, |s| \leq 1$ and $S(b) > 0$, then ${}_bm_t = \mathbb{E}({}_bz_t) < \infty, \forall t$ and the expected number of individuals younger than a at time t, started from a woman aged b at time zero, ${}_bm_t^a = \mathbb{E}({}_bz_t^a)$ satisfies

$$ {}_bm_t^a = {}_bS(t)1_{[0,a-b)}(t) + \int\limits_0^t m_{t-u}^a \; {}_b\mu(b+du), \tag{3.1} $$

where ${}_bS(t) = \frac{S(b+t)}{S(b)}$ denotes the probability a woman aged b to survive to $b+t$ and ${}_b\mu(t+b) = \frac{\mu(t+b)}{S(b)}$.

In addition if $S(t)$ and $\mu(t)$ are twice differentiable on t, then ${}_bm_t^a$ is twice differentiable both on t for $t \neq a-b$ and on b for $b \neq a$. Although derivatives don't exist in these points, there exist left and right derivatives, that are not equal to each other. If $a = \infty$ then ${}_bm_t$ is twice differentiable for all $t > 0$.

Results

Let ω be the maximum age of the life table, i.e. $\mathbb{P}(\lambda > \omega) = 0$. In other words it is the maximum age a person can reach, We can assume it is multiple of h.

Let $N_t[bh;(b+1)h)$ be the number of women in the age interval $[bh;(b+1)h)$ at time t. For example if the branching process started from one woman of age 0 at time 0 then we have $z_t = \sum_{b=0}^{\omega/h} N_t[bh;(b+1)h)$ and $N_t[bh;(b+1)h) = z_t^{bh+h} - z_t^{bh}$.

Let $C_t^a[bh;(b+1)h) = \mathbb{E} \sum_{i=1}^{N_0[bh;(b+1)h)} \eta_i z_t^a$, where $\eta_i \in [bh;(b+1)h)$ is a random variable denoting the age of woman i. We can see that $C_t^a[bh;(b+1)h)$ represents the expected number of people at time t on age less than a, who are descendants of the people on age $[bh;(b+1)h)$ at time $t = 0$. For example if the branching process started from one woman aged 0 at time 0, then $C_t^a[0;h) = m_t^a$, $C_t^a[bh;(b+1)h) = 0$ for $b \geq 1$.

Results

If we assume certain properties for the distributions of η_i we can derive the following numerical approximation.

Theorem 3

Let η_i have absolutely continuous distributions with probability density functions $f_{\eta_i}(u)$, defined on the interval $[bh; (b+1)h)$. Lets assume f_{η_i}, $S(t)$ and $\mu(t)$ are twice differentiable and $a \notin [bh, (b+1)h)$. Then the following numerical approximation holds for $h \to 0$:

$$C_t^a[bh; (b+1)h) = \mathbb{E}N_0[bh, (b+1)h)_{bh}m_t^a + O(h^3),$$

where for $t = h$ we get

$$C_h^a[bh; (b+1)h) = \mathbb{E}N_0[bh, (b+1)h)[_{bh}S(h)I\{a \geq (b+1)h\} + {}_{bh}\mu[bh; (b+1)h)] + O(h^3),$$

In addition if $\mu(t)$ is thrice differentiable, then

$$C_h^a[bh; (b+1)h) = \mathbb{E}N_0[bh, (b+1)h)[_{bh}S(h)I\{a \geq (b+1)h\} + {}_{bh}\mu'(bh)h] + O(h^3),$$

where $a \geq 0, b \geq 0$.

Results

Using Theorem 3 we can find a recursive numerical approximation of $N_t[bh,(b+1)h)$.

Theorem 4

Let η_i have twice differentiable probability density functions $f_{\eta_i}(u)$ on the interval $[bh;(b+1)h)$. Let $S(t)$ and $\mu(t)$ are thrice differentiable and $\mu[0,h]=0$. Then the following numerical approximation holds for $h\to 0$:

$$\mathbb{E}N_{t+h}[(b+1)h;(b+2)h) = \mathbb{E}N_t[bh,(b+1)h)_{bh}S(h) + O(h^3),$$

$$\mathbb{E}N_{t+h}[0;h) = \sum_{b=0}^{\omega/h}[\mathbb{E}N_t[bh,(b+1)h)_{bh}\mu'(bh)h] + O(h^2),$$

where $a\geq 0, b\geq 0$.

Results

Let ω is multiple of h and we denote $[\mathbb{E}N_t] = \begin{bmatrix} \mathbb{E}N_t[0;h) \\ \mathbb{E}N_t[h;2h) \\ \vdots \\ \mathbb{E}N_t[\omega - h;\omega) \end{bmatrix}_{(\omega/h)\times 1}$.

Let $[A] = \begin{bmatrix} {}_0\mu[0,h) & {}_h\mu[h,2h) & \cdots & {}_{\omega-h}\mu[\omega-h,\omega) \\ {}_0S(h) & 0 & \cdots & 0 \\ 0 & {}_hS(h) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {}_{\omega-h}S(h) \end{bmatrix}_{(\omega/h)\times(\omega/h)}$.

Let $[O(h^2)] = \begin{bmatrix} O(h^2) \\ O(h^2) \\ \vdots \\ O(h^2) \end{bmatrix}_{(\omega/h)\times 1}$, $[1] = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1\times(\omega/h)}$.

Results

Theorem 5

Let $N_0[0,+\infty) = N_0[b,b] = 1$, i.e. we have only one individual at time zero and she is of age b. For all $k \leq t/h$ we have

$$ {}_b m_t = [1] \cdot A^k \cdot [\mathbb{E}N_{t-kh}] + O(h). \tag{3.2} $$

Corollary 1

Substituting $h = 1, k = 1$ in Theorem 4 we get the well known method in demographics - Leslie matrix projection (see Keyfitz [4]). We see that Leslie matrix projection is actually a numerical method for solving the renewal equation for m_t. In addition we can see that the estimation error of this demography method is actually $O(h)$.

Results

Corollary 2

If we substitute $b = 0, k = t/h$ in Theorem 5 we get a numerical estimation for the expected future population count of the GBP (shown in Theorem 1).

$$m_t \approx \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} {}_0\mu[0,h) & {}_h\mu[h,2h) & \cdots & {}_{\omega-h}\mu[\omega-h,\omega) \\ {}_0S(h) & 0 & \cdots & 0 \\ 0 & {}_hS(h) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {}_{\omega-h}S(h) \end{bmatrix}^k \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Implications

- It is a very fast method for computing the solution of the renewal equation m_t.
- On every step of the method we have the expected age structure in addition to the total population count. And that doesn't costs us additional time for solving other renewal equations.
- The well-known discrete demographics method "Leslie matrix projection"is related to the theory of General Branching Processes.
- The error of Leslie matrix projection is calculated to be $O(h)$ using GBP theory.

Theorem 5. Sketch of Proof

The plan

First we show that $\mathbb{E}_\eta m_t = [1] \cdot A^k \cdot [\mathbb{E}N_{t-kh}] + O(h)$, for branching process starting from one individual aged η at time zero, where η is a random variable with twice differentiable probability density function, defined on the interval $[b, b+h)$.

Then we will show that $\mathbb{E}_\eta m_t = {}_b m_t + O(h)$, which proves that ${}_b m_t = [1] \cdot A^k \cdot [\mathbb{E}N_{t-kh}] + O(h)$. The latest is almost obvious due to the fact that ${}_u m_t$ is a smooth function of u and $\eta \in [b, b+h)$.

Theorem 5. Sketch of Proof (2)

We must prove that $A \cdot O(h^2) = \begin{bmatrix} \sum_{b=1}^{\omega/h} {}_{nh}\mu[(n-1)h; nh)O(h^2) \\ {}_0S(h)O(h^2) \\ \vdots \\ {}_{\omega-h}S(h)O(h^2) \end{bmatrix} = [O(h^2)]$.

We have that
${}_{nh}S(h) = 1 + \frac{(S(nh+h)-S(nh))}{S(nh)} = 1 + \frac{(S(nh+h)-S(nh))}{h} \cdot \frac{h}{S(nh)} \to S'(0) \cdot \frac{0}{1} = 0$.
This means that ${}_{nh}S(h)O(h^2) = O(h^3)$ when $h \to 0$.

In addition
$\sum_{n=1}^{\omega/h} {}_{nh}\mu[(n-1)h; nh)O(h^2) = \sum_{n=1}^{\omega/h} ({}_{nh}\mu'((n-1)h)h + O(h^3))O(h^2) =$
$\sum_{n=1}^{\omega/h} {}_{nh}\mu'((n-1)h)O(h^3) = \omega/h \cdot O(h^3) = O(h^2)$.

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Theorem 5. Sketch of Proof (3)

We will prove that $[\mathbb{E}N_t] = A^k \cdot [\mathbb{E}N_{t-kh}] + [O(h^2)]$ with induction. Using Theorem 4 for $k = 1$ we have that $[\mathbb{E}N_t] = A \cdot [\mathbb{E}N_{t-h}] + [O(h^2)]$ and the statement is obvious. Let's assume the statement is true for $k - 1$, i.e. $[\mathbb{E}N_t] = A^{k-1} \cdot [\mathbb{E}N_{t-(k-1)h}] + [O(h^2)]$ and prove it for k.

The functions $S(t)$ and $\mu(t)$ are smooth, so the moments of birth of new individuals have an absolutely continuous distributions. The first individual in the population at time zero has a random age, which is an absolutely continuous random variable, so at any time all individuals in the population have absolutely continuous random ages.

Theorem 5. Sketch of Proof (4)

We can apply Theorem 4 and substitute
$[\mathbb{E}N_{t-(k-1)h}] = A \cdot [\mathbb{E}N_{t-kh}] + [O(h^2)]$ inside the induction hypothesis:
$[\mathbb{E}N_t] = A^{k-1} \cdot [\mathbb{E}N_{t-(k-1)h}] + [O(h^2)] = A^k \cdot [\mathbb{E}N_{t-kh}] + A \cdot [O(h^2)] + [O(h^2)]$.
We have already proven that $A \cdot [O(h^2)] = [O(h^2)]$, so
$[\mathbb{E}N_t] = A^k \cdot [\mathbb{E}N_{t-kh}] + [O(h^2)]$, which completes the induction.

We have that $[1] \cdot [O(h^2)] = \sum\limits_{b=1}^{\omega/h} O(h^2) = \omega/hO(h^2) = O(h)$, so
$\mathbb{E}_\eta m_t = [1] \cdot [\mathbb{E}N_t] = [1] \cdot A^k \cdot [\mathbb{E}N_{t-kh}] + [O(h)]$.
We have that
$\mathbb{E}_\eta m_t = \int\limits_b^{b+h} {}_u m_t f(u) du = \int\limits_b^{b+h} ({}_b m_t + O(h)) f(u) du = {}_b m_t + O(h)$, so
${}_b m_t = [1] \cdot A^k \cdot [\mathbb{E}N_{t-kh}] + O(h)$, which completes the proof.

Application. Estimating the Malthusian Parameter

The Malthusian parameter of a population is closely related to the contribution of a live birth and can be numerically calculated as follows. From Theorem **??**

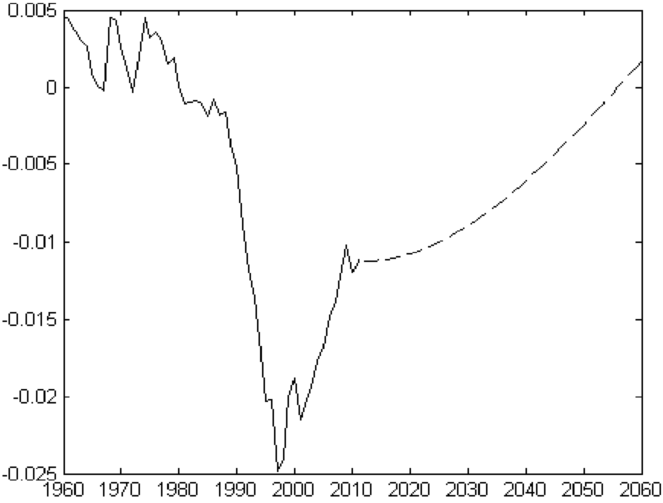
$$\frac{m_{t+h}}{m_t} \sim e^{\alpha h}, t \to \infty$$

which is the same as

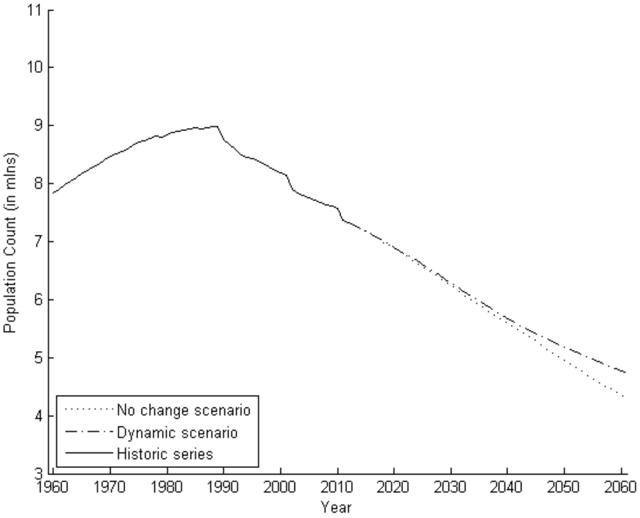
$$\frac{log(m_{t+h}) - log(m_t)}{h} \to \alpha, t \to \infty.$$

So α is approximately $(log(m_t))'$ for large t assuming m_t is smooth function.

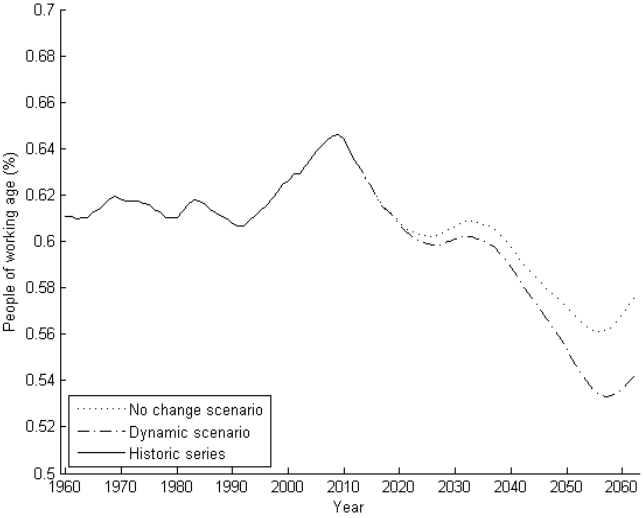
Application. Malthusian Parameter History



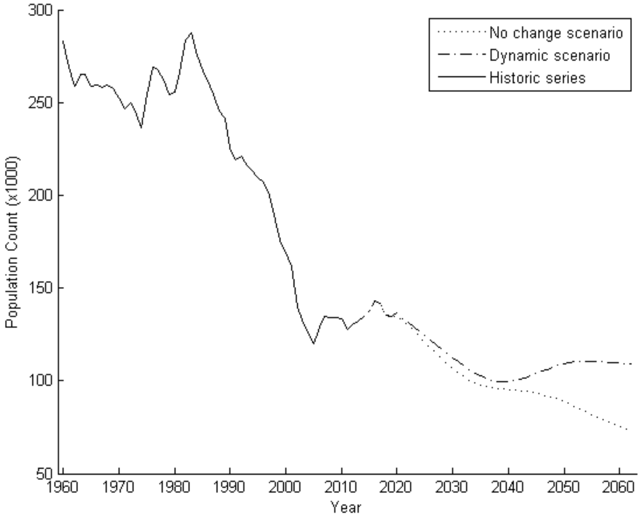
Population Count



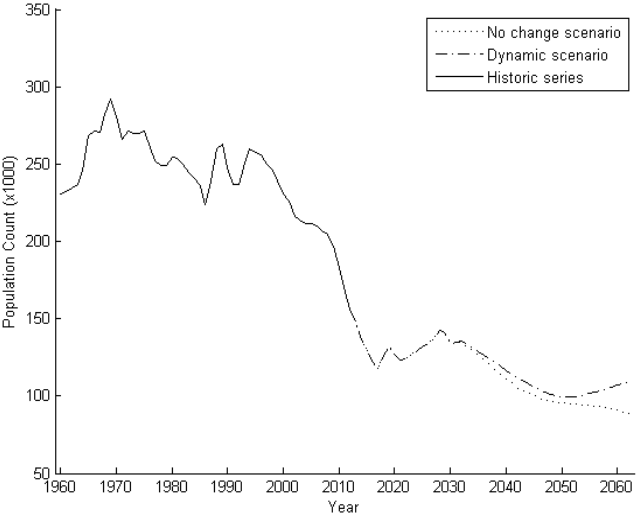
Working force



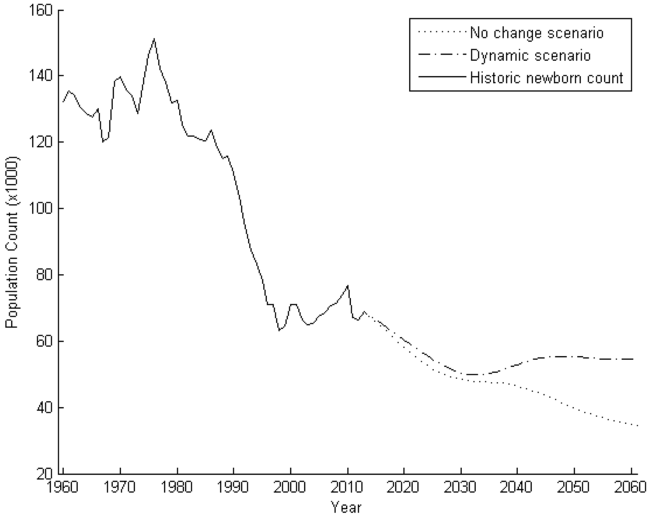
Population on school age



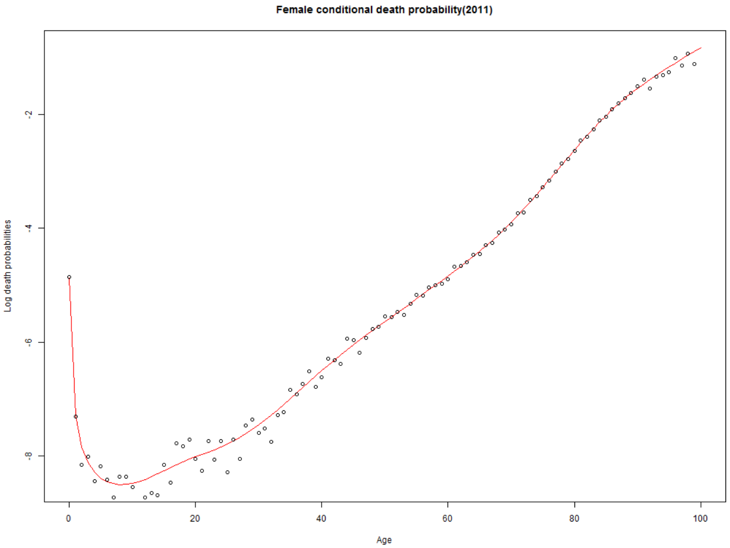
Potential Students



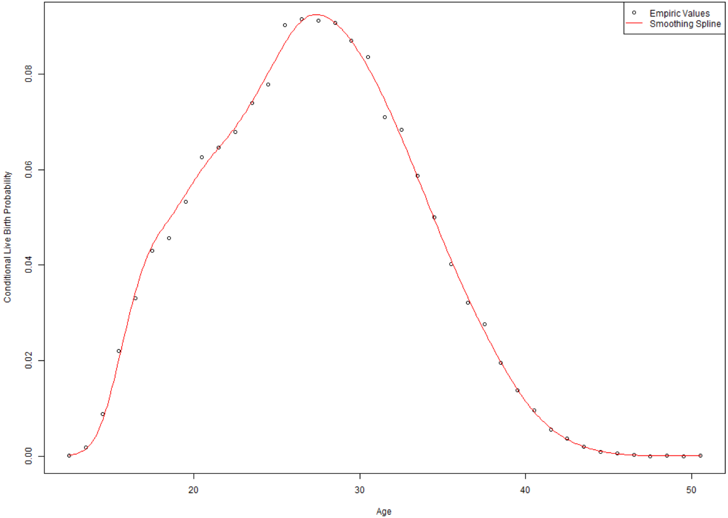
Newborns


160
140
120
100
80
60
40
20
1960
1970
1980
1990
2000
2010
2020
2030
2040
2050
2060
Year
Population Count (x1000)
No change scenario
Dynamic scenario
Historic newborn count

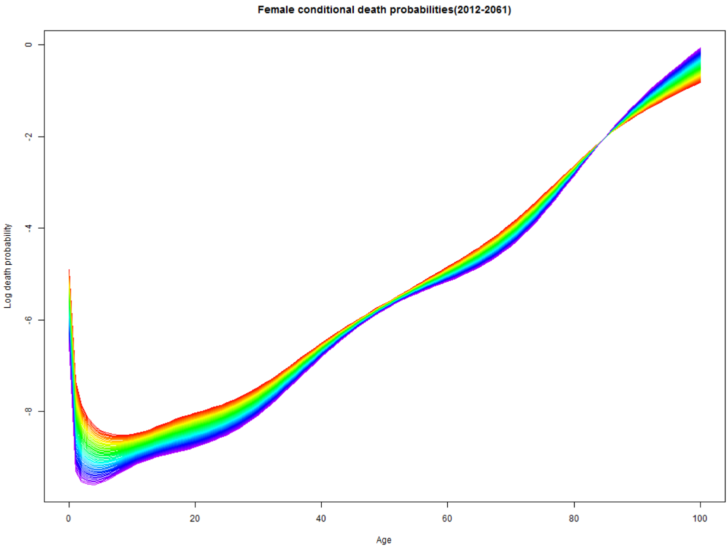
Log-Death Probabilities (Females)



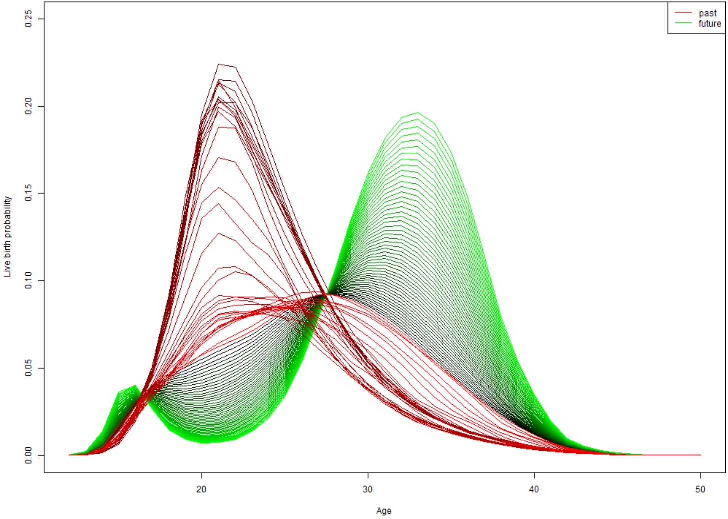
Point Process Density Function Model



Log-Death Probabilities (Females)



Point Process Density Function Forecast



The software used for calculation is R with additional packages - demography and mgcv.
The data used to model the population can be found on eurostat database website:
`http://epp.eurostat.ec.europa.eu/portal/page/portal/statistics/search_database`.

References I

- Jagers P.: Branching Processes with Biological Applications. John Wiley & Sons Ltd, (1975)
- Trayanov, P. I.: Crump–Mode–Jagers Branching Process: Modelling and Application for Human Population. Pliska Stud. Math. Bulgar. 22, 207–224 (2013)
- Slavtchova–Bojkova, M., Yanev, N. M.: Branching Stochastic Processes. St. Kliment Ohridski University press, Sofia (2007)
- Keyfitz N., Caswell H.: Applied Mathematical Demography. Springer, (2005)
- Ramsay J. O., Silverman B. W.: Functional Data Analysis. Springer, (2005)

References II

Wilmoth, J. R., Andreev, K., Jdanov, D. and Glei, D. A., "Methods Protocol for the Human Mortality Database". Unpublished Manuscript, http://www.mortality.org, (2007)

Thatcher, R., Kannisto, V. and Andreev, K., "The survivor ratio method for estimating numbers at high ages." *Demographic Research*, 6(1), 2-15. (2002).

Shkolnikov, V., "Methodology Note on the Human Life-Table Database (HLD)". Unpublished Manuscript, http://www.lifetable.de

Mode, C., *Stohastic Processes In Demography and Their Computer Implementation.* Springer, (1985).

References III

Hyndman, R., Booth, H., Tickle, L. and Maindonald, J., demography: Forecasting mortality, fertility, migration and population data. R package version 1.09-1, http://CRAN.R-project.org/package=demography, 2011.