Two types critical Bellman-Harris processes with long-lived and short-lived particles. Renewal theorems and moments increments

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Main definitions

Consider a critical Bellman-Harris branching process (BHP) $\mathbf{Z}(t) = (Z_1(t), Z_2(t))$ with two types of particles.

A particle of type $i \in \{1, 2\}$ has the life–length distribution $G_i(t)$, and, at the end of its life, produces ξ_{i1} particles of the first type and ξ_{i2} particles of the second type. The newborn particles of type $i \in \{1, 2\}$ evolve independently of each other and of the behavior of the other particles, and their evolution is stochastically equivalent to the evolution of the particles of type $i \in \{1, 2\}$ described above.

Define for $i, j, k = 1, 2$, $\delta_{ij} = 1$, if $i = j$, and $\delta_{ij} = 0$ otherwise,

$$m_{ij} := \mathbf{E}\xi_{ij},\ b^i_{jk} := \mathbf{E}\xi_{ij}\xi_{ik},\ \mathbf{M}(t) := (m_{ij}G_i(t))_{i,j=1,2},\ \mathbf{M} = \mathbf{M}(\infty).$$

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Main definitions and conditions for BHP

Assume that $\mathbf{Z}(t)$ is an irreducible, nonperiodic, critical Bellman-Harris branching process and $b^i_{jk} < \infty$. This means, in particular, that $\mathbf{M}^2 > \mathbf{0}$, the Perron root of $\mathbf{M}$ is equal to 1. $\mathbf{E}\{Z_j(t)|\mathbf{Z}(0) = (\delta_{i1}, \delta_{i2})\} = P_{ij}(t)$ for $i, j = 1, 2$.

The main conditions

$$1 - G_1(t) = o(t^{-2}) \text{ and } 1 - G_2(t) = \ell(t)t^{-\beta}, \tag{1}$$

where $\ell(t)$ is the slowly varying at infinity function as $t \to +\infty$ and $\beta \in (0, 1)$ (case $\beta = 1$ is omitted for simplicity).
Individuals at the origin in the critical catalytic branching random walk $\beta = 0, 0.5, 1$. Since 2003.
Let us presented an example of a branching Bellman-Harris process with two types of particles.

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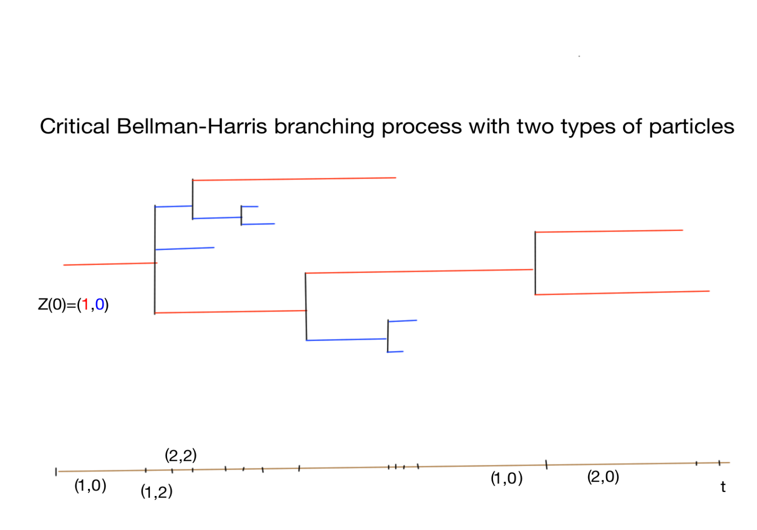
Example of BHP



Main representations

For $\mathbf{s} := (s_1, s_2) \in [0,1]^2$ and $\mathbf{z} := (z_1, z_2) \in \mathbb{Z}_+^2$ define $\mathbf{s}^{\mathbf{z}} := s_1^{z_1} s_2^{z_2}$ and denote $\mathbf{E}_i[\cdot] := \mathbf{E}[\cdot|\mathbf{Z}(0) = (\delta_{i1}, \delta_{i2})]$, $i = 1, 2$. Set

$$F_i(t;\mathbf{s}) = F_i(t; s_1, s_2) := \mathbf{E}_i \mathbf{s}^{\mathbf{Z}(t)}, \quad \mathbf{F}(t;\mathbf{s}) := (F_1(t;\mathbf{s}), F_2(t;\mathbf{s}))^T.$$

For vectors $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$ put

$$\mathbf{x} \otimes \mathbf{y} : = (x_1 y_1, x_2 y_2)^T.$$

Introduce matrices $\mathbf{G}_{\mathbf{I}}(t) := (G_i(t)\delta_{ij})_{i,j=1}^2$, $\mathbf{I}(t) := \chi_{\{t \geq 0\}}(\delta_{ij})_{i,j=1}^2$ and two–dimensional column-vectors $\mathbf{G}(t) := (G_1(t), G_2(t))^T$, $\mathbf{f}(\mathbf{s}) = (f_1(\mathbf{s}), f_2(\mathbf{s}))^T := (\mathbf{E} s_1^{\xi_{11}} s_2^{\xi_{12}}, \mathbf{E} s_1^{\xi_{21}} s_2^{\xi_{22}})^T$.

Standard total probability formula arguments leads to the system of integral equations

$$\mathbf{F}(t;\mathbf{s}) = \mathbf{s} \otimes (\mathbf{1} - \mathbf{G}(t)) + \int_0^t \mathbf{f}(\mathbf{F}(t-w, \mathbf{s})) \otimes d\mathbf{G}(w).$$

Renewal matrix

Recall that $\mathbf{G_I}(t) := (G_i(t)\delta_{ij})_{i,j=1}^2$,

$$P_{ij}(t) := \mathbf{E}_i Z_j(t) = \frac{\partial F_i(t,\mathbf{s})}{\partial s_j}\Big|_{\mathbf{s}=\mathbf{1}},\ \mathbf{P}(t) := \big(P_{ij}(t)\big)_{i,j=1}^2,$$

$$\mathbf{F}(t;\mathbf{s}) = \mathbf{s}\otimes(\mathbf{1}-\mathbf{G}(t)) + \int_0^t \mathbf{f}(\mathbf{F}(t-w,\mathbf{s}))\otimes d\mathbf{G}(w).$$

Differentiating both parts of the last equation with respect to s_1 and s_2 at $\mathbf{s}=(1,1)$ we conclude that (matrix renewal equation)

$$\mathbf{P}(t) = \mathbf{I}(t) - \mathbf{G_I}(t) + \int_0^t d\mathbf{M}(u)\mathbf{P}(t-u),\ \text{or}\ \mathbf{P}(t) = \mathbf{U}*(\mathbf{I}(\cdot)-\mathbf{G_I}(\cdot))(t),$$

were $\mathbf{U}(t) := \sum_{k=0}^{\infty}\mathbf{M}^{*k}(t)$ $\big(\mathbf{U}(t) = \mathbf{I}(t) + \int_0^t d\mathbf{M}(u)\mathbf{U}(t-u)\big)$– renewal matrix. Free term.

Renewal matrix

We define the convolution $\mathbf{C}(t) = \mathbf{A} * \mathbf{B}(t) = (C_{ij}(t))_{i,j=1}^{2}$ of two matrices $\mathbf{A}(t) = (A_{ij}(t))_{i,j=1}^{2}$ and $\mathbf{B}(t) = (B_{ij}(t))_{i,j=1}^{2}$ as the matrix with the elements

$$C_{ij}(t) := \sum\nolimits_{k=1,2} A_{ik} * B_{kj}(t).$$

In a similar way we specify the convolutions of matrices and vectors. Set $\mathbf{D} = (D_{ij})_{i,j=1}^{2}$ were

$$D_{ii} = (1 - m_{jj})/(1 - m_{11}) \text{ and } D_{ij} = m_{ij}/(1 - m_{11}), \quad i \neq j.$$

The asymptotic behavior of the renewal matrix $\mathbf{U}(t)$

Under the main conditions for some $\Gamma_\beta > 0$

$$\mathbf{U}(t) \sim \Gamma_\beta \ell^{-1}(t) t^{\beta} \mathbf{D}.$$

Increments of renewal matrix and applications

Set $\mu_1 = \int(1 - G_1(u))du < \infty$. Recall that $\mathbf{G_I}(t) := (G_i(t)\delta_{ij})_{i,j=1}^2$,

$$\mathbf{P}(t) = \mathbf{U} * (\mathbf{I}(\cdot) - \mathbf{G_I}(\cdot))(t).$$

Teugels J.L. (1968); Garsia A., Lamperti J. (1962/63); Erickson K.B. (1970-71); Doney R.A. (1997)

Theorem (Vatutin, Topchij (2013+2013(4)))

Let the main conditions are true and there exist positive constants C and T_0 such that for $t \ge T_0$ and any fixed $\Delta > 0$

$$G_2(t+\Delta) - G_2(t) \le C\Delta\ell(t)t^{\beta-1}.$$

if $\beta \in (0, 1/2]$. Then

$$\mathbf{U}(t+\Delta) - \mathbf{U}(t) \sim \Delta\beta\Gamma_\beta\ell^{-1}(t)t^{\beta-1}\mathbf{D}$$

and

$$\mathbf{P}(t) \sim \mathbf{D}\begin{pmatrix} \mu_1\beta\Gamma_\beta\ell^{-1}(t)t^{\beta-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Applications to BHP

Setting $\mathbf{Q}(t;\mathbf{s}) := \mathbf{1} - \mathbf{F}(t;\mathbf{s})$, we have

$$\mathbf{Q}(t;\mathbf{s}) = (\mathbf{1}-\mathbf{s}) \otimes (\mathbf{1} - \mathbf{G}(t)) + \int_0^t (\mathbf{1} - \mathbf{f}(\mathbf{F}(t-w;\mathbf{s}))) \otimes d\mathbf{G}(w).$$

Letting further

$$\mathbf{G}_{\mathbf{I}}(t) := (G_i(t)\delta_{ij})_{i,j=1}^2.$$

$$\vec{\Phi}(\mathbf{s}) = (\Phi_1(\mathbf{s}), \Phi_2(\mathbf{s}))^\dagger := \mathbf{M}\mathbf{s} - (\mathbf{1} - \mathbf{f}(\mathbf{1}-\mathbf{s}))$$

we get (free term)

$$\begin{aligned}\mathbf{Q}(t;\mathbf{s}) &= (\mathbf{1}-\mathbf{s}) \otimes (\mathbf{1} - \mathbf{G}(t)) + \int_0^t d\mathbf{M}(w)\mathbf{Q}(t-w;\mathbf{s}) \\ &- \int_0^t d\mathbf{G}_{\mathbf{I}}(w)\vec{\Phi}(\mathbf{Q}(t-w;\mathbf{s})).\end{aligned}$$

Applications to BHP

or, applying the renewal theorem

$$\mathbf{Q}(t;\mathbf{s}) = \mathbf{U}*((\mathbf{1}-\mathbf{s})\otimes(\mathbf{1}-\mathbf{G}(\cdot)))(t)-\int_0^t d(\mathbf{U}*\mathbf{G}_\mathbf{I})(w)\vec{\Phi}(\mathbf{Q}(t-w;\mathbf{s})).$$

The values of the components of $\mathbf{Q}(t;\mathbf{s})$ have different orders and require different normalization. It means that two expression in right side in this representation have one and the same asymptotic and increments of $\mathbf{U}(t)$ (as well as for $\mathbf{P}(t)$) are very useful to investigation of the last difference.

Some simple theorems based on this representations, was obtained in joint work with V.A. Vatutin (2013) and V.A. Vatutin & A.M. Iksanov (2013). $[1-s_i(t)\to 0]$ We hope to continue these investigations.

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Examples

Let the main conditions are true, then

$$\mathbf{P}_i(\mathbf{Z}(t) \neq \mathbf{0}) \sim \mathbf{P}_i(Z_2(t) > 0) \sim c\sqrt{1 - G_2(t)},$$
$$\mathbf{P}_i(Z_1(t) > 0) = o(\mathbf{P}_i(Z_2(t) > 0))$$

$\beta \in (0, 0.5]$ and $\int_0^\infty (1 - G_2(t))^2 dt < \infty$ + VT 2013

$$\mathbf{P}_i(Z_1(t) > 0) \sim c(t(1 - G_2(t))^{-1}, \ \mathbf{E}_i\left[s^{Z_1(t)} | Z_1(t) > 0\right] \to \phi_i(s).$$

$\beta = 0.5$ and $\ell_1(t) := \int_0^t (1 - G_2(t))^2 dt \to \infty$, Hypothesis C ?

$$\mathbf{P}_i(Z_1(t) > 0) \sim c(\ell_1(t) t (1 - G_2(t))^{-1}.$$

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Renewal matrix

The main subject of our investigation is the renewal matrix

$$\mathbf{U}(t) := \sum_{k=0}^{\infty} \mathbf{M}^{*k}(t).$$

Define the distribution functions

$$\begin{aligned} H_1(t) &:= (1-m_{11})\sum_{s=0}^{\infty} m_{11}^s G_1^{*s}(t), \\ X_1(t) &= m_{22}G_2(t) + (1-m_{22})G_1 * G_2 * H_1(t). \end{aligned}$$

and matrices $\mathbf{U}^c(t) := \mathbf{U}(t) - \mathbf{U}^d(t)$, were

$$\mathbf{U}^d(t) = (U_{ij}^d(t))_{i,j=1}^2 := \begin{pmatrix} \frac{H_1(t)}{1-m_{11}} & \frac{m_{12}G_1 * H_1(t)}{1-m_{11}} \\ 0 & 0 \end{pmatrix}.$$

Components of renewal matrix

The distribution $H_1(t)$ has the estimations of the tail and increments at infinity of the same order as $G_1(t)$. Therefore, we choose the restrictions on $G_1(t)$, providing no effect of $\mathbf{U}^d(t)$ on asymptotic results. $\Delta H_1(t) = o(yt^{-\beta-2}\ell(t))$

The matrix $\mathbf{U}^c(t)$ is the main part of $\mathbf{U}(t)$ and has the form

$$U^c_{ij}(t) := V_{ij} * U_{X_1}(t), \ \ i,j = 1,2,$$

were $U_{X_1}(t) := \sum_{k=0}^{\infty} X_1^{*k}(t)$ and

$$V_{11}(t) := \frac{(X_1(\cdot) - m_{22}G_2(\cdot)) * H_1(t)}{1-m_{11}}, \quad V_{12}(t) := \frac{m_{12}G_1 * X_1 * H_1(t)}{1-m_{11}},$$

$$V_{21}(t) := \frac{m_{21}G_2 * H_1(t)}{1-m_{11}}, \qquad V_{22}(t) := 1.$$

Note that $X_1(t)$, $U_{X_1}(t)$ and $V_{11}(t)$ are contain the component $G_2(t)$ and smoothness of $G_2(t)$ are transferred to all of these functions.

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Hypothesis C

For $\gamma \in (0,1]$ define $\mathcal{N}_\gamma(t)$ as

$$\mathcal{N}_\gamma^{-(\beta+2)\gamma^{-1}}(t)\ell\big(\mathcal{N}_\gamma^{\gamma^{-1}}(t)\big) \sim t^{-\beta}\ell(t),$$

it means that

$$\mathcal{N}_\gamma(t) = t^{\beta\gamma(\beta+2)^{-1}}\ell_{\mathcal{N}_\gamma}(t).$$

Smoothness conditions for $\beta \in (0.5, 1)$, $\beta \in (0, 0.5]$

Let $\exists\ \gamma \in (0, \min\{1, (1-\beta)(\beta+2)/\beta\}]$ such that for $y \in (1, 10t^{\beta\gamma}]$ $(1 - G_1(t) = o(t^{-2-\beta-\delta}))$

$$G_1(t) - G_1(t - y\mathcal{N}_\gamma(t)) = o(yt^{-\beta-2}\ell(t)), \quad (< Kyt^{-\beta-2}\ell(t))$$

$g_2'(t)$ absolutely continuous (monotonous), $|g_2''(t)|$ integrable,

$$g_2'(t) \sim -(\beta+1)\beta t^{-\beta-2}\ell(t) \quad (< Kt^{-\beta-2}\ell(t))$$

and for some $C \geq 1$

$$\operatorname*{Var}_{u>t} u^2 g_2'(u) \sim C(\beta+1)\beta t^{-\beta}\ell(t);$$

Renewal density Derivative for $\mathbf{U}^c(t)$ and increments for $\mathbf{U}^d(t)$

Theorem

Let Hypothesis C be valid. Then the density for renewal matrix $\mathbf{u}^c(t)$ is absolutely continuous and

$$(\mathbf{u}^c(t))' \sim (\beta - 1)\beta\Gamma_\beta \ell^{-1}(t) t^{\beta-2}\mathbf{D},$$

more of then $i = 1, 2$

$$U_{i1}^d(t) - U_{i1}^d(t - \mathcal{N}_\gamma(t)) = \begin{cases} O(t^{-\beta-2}\ell(t)), & \text{if} \quad \beta \in (0, 0.5], \\ o(t^{-\beta-2}\ell(t)), & \text{if} \quad \beta \in (0.5, 1). \end{cases}$$

Applications

Define

$$\mathbf{P}^{dc}(t) = \left(P_{ij}^{dc}(t)\right)_{i,j=1}^{2} := \mathbf{U}^{dc} * (\mathbf{I} - \mathbf{G}_{\mathbf{I}}(\cdot))(t).$$

Theorem

Let Hypothesis C be valid. Then the functions $P_{i1}(t)$, $i = 1, 2$, are monotone for sufficiently large t and the estimations

$$P_{i1}^{d}(t) - P_{i1}^{d}(t - \mathcal{N}_{\gamma}(t)) = \begin{cases} O(t^{-\beta-2}\ell(t)), & \text{if} \quad \beta \in (0, 0.5], \\ o(t^{-\beta-2}\ell(t)), & \text{if} \quad \beta \in (0.5, 1), \end{cases}$$

are true.
The functions $P_{ij}^{c}(t)$, $i, j = 1, 2$, are absolutely continuous and

$$P_{i1}^{c}{}'(t) \sim D_{i1}\mu_1\beta(\beta-1)\Gamma_{\beta}\ell^{-1}(t)t^{\beta-2}, \; P_{i2}^{c}{}'(t) = o(t^{-1}).$$

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Thank you very much!

Muchas gracias!

Огромное спасибо!