Statistical inference for 2-type doubly symmetric critical irreducible CBI processes

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Galton–Watson branching process with immigration

GWI process:

$$\zeta_k = \sum_{j=1}^{\zeta_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \mathbb{N} := \{1, 2, \ldots\},$$

$\{\xi_{k,j}, \varepsilon_k : k, j \in \mathbb{N}\}$ independent rv's with values in $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$

$\{\xi_{k,j} : k, j \in \mathbb{N}\}$ identically distributed

$\{\varepsilon_k : k \in \mathbb{N}\}$ identically distributed

Possible scaling limits: CBI processes (Kawazu and Watanabe, 1971 TVP; Li, 2006 JAP)

$\forall n \in \mathbb{N}$, let $(\zeta_k^{(n)})_{k\in\mathbb{Z}_+}$ be a GWI process, and $\gamma_n \in \mathbb{R}_{++}$ with $\gamma_n \uparrow \infty$. Under certain conditions, $(n^{-1}\zeta_{\lfloor \gamma_n t\rfloor}^{(n)})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (X_t)_{t\in\mathbb{R}_+}$ as $n \to \infty$, where $(X_t)_{t\in\mathbb{R}_+}$ is a conservative time-homogeneous Markov process with state space $\mathbb{R}_+$ and with infinitesimal generator

$$\begin{aligned}(\mathcal{A}f)(x) = (bx+\beta)f'(x) + cxf''(x) + \int_0^\infty \big[f(x+z) - f(x)\big]\,\nu(\mathrm{d}z)\\ + x\int_0^\infty \big[f(x+z) - f(x) - f'(x)(1\wedge z)\big]\,\mu(\mathrm{d}z)\end{aligned}$$

for $f \in \mathbb{C}_\mathrm{c}^2(\mathbb{R}_+,\mathbb{R})$ and $x \in \mathbb{R}_+$, where $b \in \mathbb{R}$, $\beta, c \in \mathbb{R}_+$, and ν, μ are Borel measures on $(0,\infty)$ with $\int_0^\infty (1\wedge z)\,\nu(\mathrm{d}z) < \infty$ and $\int_0^\infty (z\wedge z^2)\,\mu(\mathrm{d}z) < \infty$.

The Markov process $(X_t)_{t\in\mathbb{R}_+}$ is called a CBI process with parameter vector (b, c, μ, β, ν).

SDE of a single-type CBI process (Dawson and Li, 2006 AP)

If $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$ then there is a pathwise unique non-negative strong solution to SDE

$$X_t = X_0 + \int_0^t (\widetilde{b}X_s + \beta)\,\mathrm{d}s + \int_0^t \sqrt{2cX_s^+}\,\mathrm{d}W_s + \int_0^t \int_0^\infty \int_0^{X_{s-}} z\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \int_0^t \int_0^\infty z\,M(\mathrm{d}s,\mathrm{d}z), \qquad t \in \mathbb{R}_+,$$

where

- $\widetilde{b} := b + \int_1^\infty (z-1)\,\mu(\mathrm{d}z)$,
- $(W_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process,
- N and M are Poisson random measures on $\mathbb{R}^3_{++}$ and $\mathbb{R}^2_{++}$ with intensity measures $\mathrm{d}s\,\mu(\mathrm{d}z)\,\mathrm{d}u$ and $\mathrm{d}s\,\nu(\mathrm{d}z)$,
- $\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) := N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) - \mathrm{d}s\,\mu(\mathrm{d}z)\,\mathrm{d}u$,
- $(W_t)_{t\in\mathbb{R}_+}$, N and M are independent,

and the solution is a CBI process with parameter vector (b, c, μ, β, ν).

Expectation of a CBI(B, c, μ, β, ν) process if $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$

$$\mathbb{E}(X_t \mid X_0 = x) = \mathrm{e}^{\widetilde{b}t} x + \widetilde{\beta} \int_0^t \mathrm{e}^{\widetilde{b}u}\,\mathrm{d}u, \qquad x \in \mathbb{R}_+, \quad t \in \mathbb{R}_+,$$

with $\widetilde{\beta} := \beta + \int_0^\infty z\,\nu(\mathrm{d}z)$.

Interpretation of $\mathrm{e}^{\widetilde{b}}$: branching mean

$$\mathrm{e}^{\widetilde{b}} = \mathbb{E}(Y_1 \mid Y_0 = 1),$$

where $(Y_t)_{t \in \mathbb{R}_+}$ is a CBI$(b, c, \mu, 0, 0)$ process, which can be considered as a pure branching process (without immigration).

$-\widetilde{b}$ can also be considered as the death rate

Interpretation of $\widetilde{\beta}$: immigration mean

$$\widetilde{\beta} = \mathbb{E}(Z_1 \mid Z_0 = 0),$$

where $(Z_t)_{t \in \mathbb{R}_+}$ is a CBI$(0, 0, 0, \beta, \nu)$ process, which can be considered as a pure immigration process (without branching).

Asymptotics of the expectation if $\int_1^\infty z\,\nu(\mathrm{d}z) < \infty$

- $\lim\limits_{t\to\infty} \mathbb{E}(X_t \,|\, X_0 = x) = -\frac{\widetilde{\beta}}{\widetilde{b}}$ if $\widetilde{b} < 0$ (subcritical case);
- $\lim\limits_{t\to\infty} t^{-1}\, \mathbb{E}(X_t \,|\, X_0 = x) = \widetilde{\beta}$ if $\widetilde{b} = 0$ (critical case);
- $\lim\limits_{t\to\infty} \mathrm{e}^{-\widetilde{b}t}\, \mathbb{E}(X_t \,|\, X_0 = x) = x + \frac{\widetilde{\beta}}{\widetilde{b}}$ if $\widetilde{b} > 0$ (supercritical case).

- Overbeck and Rydén (1997 ET): CBI$(b, c, 0, \beta, 0)$ = CIR = Cox–Ingersol–Ross stochastic volatility model; asymptotic normality of CLSE of $(\widetilde{b}, c, \widetilde{\beta})$ in the subcritical case
- Huang, Ma and Zhu (2011 SPL): CBI(b, c, μ, β, ν) process with $\int_1^\infty z^2\,\mu(\mathrm{d}z) + \int_1^\infty z^2\,\nu(\mathrm{d}z) < \infty$; weighted CLSE of $(\widetilde{b}, \widetilde{\beta})$ in the subcritical, critical and supercritical cases
- Li and Ma (2015 SPA): stable CIR model driven by a spectrally positive α-stable process with index $\alpha \in (1, 2)$, which is a special CBI$(b, 0, \mu, \beta, 0)$ process with $\mu(\mathrm{d}z) = cz^{-\alpha-1}\,\mathrm{d}z$ on $(0, \infty)$; CLSE and weighted CLSE of $(\widetilde{b}, \widetilde{\beta})$ in the subcritical case

Conditional least squares estimator (CLSE)

Martingale differences:

$$M_k := X_k - \mathbb{E}(X_k \mid \mathcal{F}_{k-1}^X) = X_k - \varrho X_{k-1} - \overline{\beta}, \qquad k \in \mathbb{N},$$

where $\varrho := \mathrm{e}^{\widetilde{b}}$ and $\overline{\beta} := \widetilde{\beta} \int_0^1 \mathrm{e}^{\widetilde{b}u} \, \mathrm{d}u$.

Autoregression with non-negative drift: $X_k = \varrho X_{k-1} + \overline{\beta} + M_k$, $k \in \mathbb{N}$.
CLSE of ϱ and $\overline{\beta}$ based on observations $X_0, X_1, \ldots, X_n$:

$$\begin{bmatrix} \widehat{\varrho}_n \\ \widehat{\overline{\beta}}_n \end{bmatrix} := \operatorname*{arg\,min}_{(\varrho, \overline{\beta}) \in \mathbb{R}^2} \sum_{k=1}^n (X_k - \varrho X_{k-1} - \overline{\beta})^2$$

$$= \frac{1}{n \sum\limits_{k=1}^n X_{k-1}^2 - \left(\sum\limits_{k=1}^n X_{k-1} \right)^2} \begin{bmatrix} n \sum\limits_{k=1}^n X_k X_{k-1} - \sum\limits_{k=1}^n X_k \sum\limits_{k=1}^n X_{k-1} \\ \sum\limits_{k=1}^n X_k \sum\limits_{k=1}^n X_{k-1}^2 - \sum\limits_{k=1}^n X_k X_{k-1} \sum\limits_{k=1}^n X_{k-1} \end{bmatrix},$$

provided that $n \sum\limits_{k=1}^n X_{k-1}^2 - \left(\sum\limits_{k=1}^n X_{k-1} \right)^2 > 0$.

We have

$$(\varrho, \overline{\beta}) = \left(\mathrm{e}^{\widetilde{b}}, \widetilde{\beta} \int_0^1 \mathrm{e}^{\widetilde{b}s} \, \mathrm{d}s\right) =: h(\widetilde{b}, \widetilde{\beta}), \qquad (\widetilde{b}, \widetilde{\beta}) \in \mathbb{R}^2,$$

hence the CLSE of $(\widetilde{b}, \widetilde{\beta})$:

$$(\widehat{\widetilde{b}}_n, \widehat{\widetilde{\beta}}_n) = h^{-1}(\widehat{\varrho}_n, \widehat{\overline{\beta}}_n) = \left(\log(\widehat{\varrho}_n), \frac{\widehat{\overline{\beta}}_n}{\int_0^1 (\widehat{\varrho}_n)^s \, \mathrm{d}s}\right), \qquad n \in \mathbb{N},$$

provided $\widehat{\varrho}_n > 0$.

Asymptotics of CLSE : $\widetilde{b} = 0$ (Barczy, Körmendi and P, 2014)

If $X_0 = 0$, $\int_1^\infty z^8 \, \mu(\mathrm{d}z) < \infty$, $\int_1^\infty z^8 \, \nu(\mathrm{d}z) < \infty$ and $\widetilde{\beta} > 0$ then

$$\begin{bmatrix} n(\widehat{\widetilde{b}}_n - \widetilde{b}) \\ \widehat{\widetilde{\beta}}_n - \widetilde{\beta} \end{bmatrix} \xrightarrow{\mathcal{D}} \frac{1}{\int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t - \left(\int_0^1 \mathcal{Y}_t \, \mathrm{d}t\right)^2} \begin{bmatrix} \int_0^1 \mathcal{Y}_t \, \mathrm{d}\mathcal{M}_t - \mathcal{M}_1 \int_0^1 \mathcal{Y}_t \, \mathrm{d}t \\ \mathcal{M}_1 \int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t - \int_0^1 \mathcal{Y}_t \, \mathrm{d}t \int_0^1 \mathcal{Y}_t \, \mathrm{d}\mathcal{M}_t \end{bmatrix}$$

where $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\mathrm{d}\mathcal{Y}_t = \widetilde{\beta} \, \mathrm{d}t + \sqrt{C \mathcal{Y}_t^+} \, \mathrm{d}\mathcal{W}_t, \quad t \in \mathbb{R}_+, \qquad \mathcal{Y}_0 = 0,$$

where $C := 2c + \int_0^\infty z^2 \, \mu(\mathrm{d}z)$ and $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, and $\mathcal{M}_t := \mathcal{Y}_t - \widetilde{\beta} t$, $t \in \mathbb{R}_+$.

If, in addition, $c = 0$ and $\mu = 0$ (hence the process is a pure immigration process), then

$$\begin{bmatrix} n^{3/2}(\widehat{\widetilde{b}}_n - \widetilde{b}) \\ n^{1/2}(\widehat{\widetilde{\beta}}_n - \widetilde{\beta}) \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2 \left(\mathbf{0}, \int_0^\infty z^2 \, \nu(\mathrm{d}z) \begin{bmatrix} \frac{1}{3}(\widetilde{\beta})^2 & \frac{1}{2}\widetilde{\beta} \\ \frac{1}{2}\widetilde{\beta} & 1 \end{bmatrix}^{-1} \right).$$

We have $C = \operatorname{Var}(Y_1 \mid Y_0 = 1)$, where $(Y_t)_{t \in \mathbb{R}_+}$ is a pure branching process $\mathrm{CBI}(B, c, \mu, 0, 0)$, hence C is the branching variance.

Multi-type CBI process with parameter $(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$

Conservative time-homogeneous Markov process $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ with state space $\mathbb{R}_+^d$ and with infinitesimal generator

$$(\mathcal{A}f)(\boldsymbol{x}) = \langle \boldsymbol{\beta} + \boldsymbol{B}\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x})\rangle + \sum_{i=1}^{d} c_i x_i f''_{i,i}(\boldsymbol{x}) + \int_{\mathcal{U}_d} \big[f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x})\big]\,\nu(\mathrm{d}\boldsymbol{z})$$

$$+ \sum_{i=1}^{d} x_i \int_{\mathcal{U}_d} \big[f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x}) - f'_i(\boldsymbol{x})(1\wedge z_i)\big]\,\mu_i(\mathrm{d}\boldsymbol{z})$$

for $f \in \mathbb{C}_c^2(\mathbb{R}_+^d, \mathbb{R})$ and $\boldsymbol{x} \in \mathbb{R}_+^d$, where $\boldsymbol{B} \in \mathbb{R}_{(+)}^{d\times d}$, $\boldsymbol{\beta}, \boldsymbol{c} \in \mathbb{R}_+^d$, ν is a Borel measure on $\mathcal{U}_d := \mathbb{R}_+^d \setminus \{\boldsymbol{0}\}$ satisfying $\int_{\mathcal{U}_d} (1 \wedge \|\boldsymbol{z}\|)\,\nu(\mathrm{d}\boldsymbol{z}) < \infty$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$, where, for each $i \in \{1, \ldots, d\}$, μ_i is a Borel measure on $\mathcal{U}_d$ satisfying

$$\int_{\mathcal{U}_d} \left[(\|\boldsymbol{z}\| \wedge \|\boldsymbol{z}\|^2) + \sum_{j\in\{1,\ldots,d\}\setminus\{i\}} (1 \wedge z_j)\right] \mu_i(\mathrm{d}\boldsymbol{z}) < \infty.$$

SDE of a MCBI process (Barczy, Li and P, 2015 ALEA)

If $\int_{\mathcal{U}_d} \|\boldsymbol{z}\| \, \nu(\mathrm{d}z) < \infty$ then there is a unique non-negative strong solution to SDE

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t (\widetilde{\boldsymbol{B}} \boldsymbol{X}_s + \boldsymbol{\beta}) \, \mathrm{d}s + \sum_{i=1}^d \boldsymbol{e}_i \int_0^t \sqrt{2c_i X_{s,i}^+} \, \mathrm{d}W_{s,i}$$
$$+ \sum_{j=1}^d \int_0^t \int_{\mathcal{U}_d} \int_0^{X_{s-,j}} \boldsymbol{z} \, \widetilde{N}_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) + \int_0^t \int_{\mathcal{U}_d} \boldsymbol{z} \, M(\mathrm{d}s, \mathrm{d}\boldsymbol{z}), \qquad t \in \mathbb{R}_+,$$

where

- $\widetilde{\boldsymbol{B}} := (\widetilde{b}_{i,j})_{i,j \in \{1,\ldots,d\}} \in \mathbb{R}_{(+)}^{d \times d}$, $\widetilde{b}_{i,j} := b_{i,j} + \int_{\mathcal{U}_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z})$,
- $(\boldsymbol{W}_t)_{t \in \mathbb{R}_+}$ is a d-dimensional standard Wiener process,
- $N_1, \ldots, N_d$ and M are Poisson random measures on $\mathbb{R}_{++} \times \mathcal{U}_d \times \mathbb{R}_{++}$ and $\mathbb{R}_{++} \times \mathcal{U}_d$ with intensity measures $\mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u$ and $\mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{z})$,
- $\widetilde{N}_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) := N_j(\mathrm{d}s, \mathrm{d}\boldsymbol{z}, \mathrm{d}u) - \mathrm{d}s \, \mu(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u$, $j \in \{1, \ldots, d\}$,
- $(\boldsymbol{W}_t)_{t \in \mathbb{R}_+}$, $N_1, \ldots, N_d$ and M are independent,

and the solution is a CBI process with parameter $(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$.

Expectation of an MCBI($d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu$) process

$$\mathbb{E}(\boldsymbol{X}_t \mid \boldsymbol{X}_0 = \boldsymbol{x}) = \mathrm{e}^{t\widetilde{\boldsymbol{B}}}\boldsymbol{x} + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}u, \qquad \boldsymbol{x} \in \mathbb{R}_+^d, \quad t \in \mathbb{R}_+,$$

with $\widetilde{\boldsymbol{\beta}} := \boldsymbol{\beta} + \int_{\mathcal{U}_d} \boldsymbol{z}\,\nu(\mathrm{d}\boldsymbol{z})$.

Interpretation of $\mathrm{e}^{\boldsymbol{B}}$: branching mean matrix

$$\mathrm{e}^{\widetilde{\boldsymbol{B}}}\boldsymbol{e}_j = \mathbb{E}(\boldsymbol{Y}_1 \mid \boldsymbol{Y}_0 = \boldsymbol{e}_j), \qquad j \in \{1, \ldots, d\},$$

where $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ is an MCBI($d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{0}, 0$) process, which can be considered as a pure branching process (without immigration).

Interpretation of $\boldsymbol{\beta}$: immigration mean vector

$$\widetilde{\boldsymbol{\beta}} = \mathbb{E}(\boldsymbol{Z}_1 \mid \boldsymbol{Z}_0 = \boldsymbol{0}),$$

where $(\boldsymbol{Z}_t)_{t\in\mathbb{R}_+}$ is an MCBI($d, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\beta}, \nu$) process, which can be considered as a pure immigration process (without branching).

Irreducibility of a matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$

A matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ is called reducible if there exist a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{d\times d}$ and an integer r with $1 \leqslant r \leqslant d-1$ such that

$$\boldsymbol{P}^\top \boldsymbol{A} \boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{0} & \boldsymbol{A}_3 \end{bmatrix},$$

where $\boldsymbol{A}_1 \in \mathbb{R}^{r\times r}$, $\boldsymbol{A}_3 \in \mathbb{R}^{(d-r)\times(d-r)}$, $\boldsymbol{A}_2 \in \mathbb{R}^{r\times(d-r)}$, and $\boldsymbol{0} \in \mathbb{R}^{(d-r)\times r}$ is a null matrix. A matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ is called irreducible if it is not reducible. (Hence 1-by-1 matrices are irreducible.)

$e^{t\widetilde{\boldsymbol{B}}} \in \mathbb{R}_+^{d\times d}$ for all $t \in \mathbb{R}_+$.

The following statements are equivalent:

- $\exists\ t_0 \in \mathbb{R}_{++} := (0,\infty)$ with $e^{t_0\widetilde{\boldsymbol{B}}} \in \mathbb{R}_{++}^{d\times d}$;
- $\forall\ t \in \mathbb{R}_{++}$ we have $e^{t\widetilde{\boldsymbol{B}}} \in \mathbb{R}_{++}^{d\times d}$;
- $e^{\widetilde{\boldsymbol{B}}}$ is irreducible;
- $\widetilde{\boldsymbol{B}}$ is irreducible.

Irreducibility

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be an MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \beta, \nu)$. Then $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is called irreducible if $\widetilde{\boldsymbol{B}}$ is irreducible.

For a matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$, put

$$\sigma(\boldsymbol{A}) := \text{set of the eigenvalues of } \boldsymbol{A},$$

$$r(\boldsymbol{A}) := \max_{\lambda \in \sigma(\boldsymbol{A})} |\lambda| \qquad \text{(spectral radius of } \boldsymbol{A}),$$

$$s(\boldsymbol{A}) := \max_{\lambda \in \sigma(\boldsymbol{A})} \operatorname{Re}(\lambda) = \log r(\mathrm{e}^{\boldsymbol{A}}) \qquad \text{(by spectral mapping theorem).}$$

Asymptotics of the expectation

- $\lim_{t \to \infty} \mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = -\widetilde{\boldsymbol{B}}^{-1} \widetilde{\beta}$ if $s(\widetilde{\boldsymbol{B}}) < 0$ (subcritical case);
- $\lim_{t \to \infty} t^{-1} \mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = \boldsymbol{\Pi} \widetilde{\beta}$ if $s(\widetilde{\boldsymbol{B}}) = 0$ (critical case);
- $\lim_{t \to \infty} \mathrm{e}^{-s(\widetilde{\boldsymbol{B}})t} \mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = \boldsymbol{\Pi} \boldsymbol{x} + \frac{1}{s(\widetilde{\boldsymbol{B}})} \boldsymbol{\Pi} \widetilde{\beta}$ if $s(\widetilde{\boldsymbol{B}}) > 0$ (supercritical case),

with $\boldsymbol{\Pi} := \widetilde{\boldsymbol{u}} \boldsymbol{u}^\top \in \mathbb{R}_{++}^{d \times d}$, where $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{u}$ are the right and left Perron eigenvectors of $\widetilde{\boldsymbol{B}}$, corresponding to the eigenvalue $s(\widetilde{\boldsymbol{B}})$.

- Xu (2014): special irreducible and subcritical MCBI$(2, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{0}, \beta, 0)$ diffusion processes; asymptotic normality of CLS and weighted CLS estimators of $(\widetilde{\boldsymbol{B}}, \boldsymbol{c}, \widetilde{\beta})$

Limit theorem for a discretized irreducible and critical multi-type CBI process (Barczy and P, 2014)

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be an irreducible and critical MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{\beta}, \nu)$ process with some moment conditions. Then

$$(\boldsymbol{\mathcal{X}}_t^{(n)})_{t\in\mathbb{R}_+} := (n^{-1}\boldsymbol{X}_{\lfloor nt\rfloor})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{\mathcal{X}}_t)_{t\in\mathbb{R}_+} := (\mathcal{X}_t\widetilde{\boldsymbol{u}})_{t\in\mathbb{R}_+}$$

as $n\to\infty$, where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is the unique strong solution of the SDE

$$\mathrm{d}\mathcal{X}_t = \langle \boldsymbol{u}, \boldsymbol{\beta}\rangle\,\mathrm{d}t + \sqrt{\langle \widetilde{\boldsymbol{C}}\boldsymbol{u}, \boldsymbol{u}\rangle \mathcal{X}_t^+}\,\mathrm{d}\mathcal{W}_t, \quad t\in\mathbb{R}_+, \qquad \mathcal{X}_0 = 0,$$

where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process and

$$\widetilde{\boldsymbol{C}} := \mathrm{Var}(\boldsymbol{Y}_1 \mid \boldsymbol{Y}_0 = \widetilde{\boldsymbol{u}}) \in \mathbb{R}_+^{d\times d},$$

where $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ is an MCBI$(d, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{\mu}, \boldsymbol{0}, 0)$ (pure branching) process.

Doubly symmetric CBI process

$\widetilde{\boldsymbol{B}} = \begin{bmatrix} \gamma & \kappa \\ \kappa & \gamma \end{bmatrix}$ with some $\gamma \in \mathbb{R}$ and $\kappa \in \mathbb{R}_+$.

$-\gamma$: death rate of both types

κ : transformation rate of one type to the other type

$\widetilde{\boldsymbol{B}}$ is irreducible if and only if $\kappa > 0$

Eigenvalues of $\widetilde{\boldsymbol{B}}$ are $s := \gamma + \kappa$ (criticality parameter) and $\gamma - \kappa$,

$$\widetilde{\boldsymbol{u}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$\widetilde{\boldsymbol{B}}$ is critical if and only if $s = 0$

Conditional least squares estimator (CLSE)

Martingale differences:

$$\boldsymbol{M}_k := \boldsymbol{X}_k - \mathbb{E}(\boldsymbol{X}_k \mid \mathcal{F}_{k-1}^X) = \boldsymbol{X}_k - \mathrm{e}^{\widetilde{\boldsymbol{B}}} X_{k-1} - \overline{\boldsymbol{\beta}}, \qquad k \in \mathbb{N},$$

with $\overline{\boldsymbol{\beta}} := \int_0^1 \mathrm{e}^{u\boldsymbol{B}} \boldsymbol{\beta} \, \mathrm{d}u$. Autoregression with non-negative drift:

$$\boldsymbol{X}_k = \mathrm{e}^{\widetilde{\boldsymbol{B}}} X_{k-1} + \overline{\boldsymbol{\beta}} + \boldsymbol{M}_k, \quad k \in \mathbb{N},$$

where

$$\mathrm{e}^{\widetilde{\boldsymbol{B}}} = \mathrm{e}^{\gamma} \begin{bmatrix} \cosh(\kappa) & \sinh(\kappa) \\ \sinh(\kappa) & \cosh(\kappa) \end{bmatrix}.$$

Putting

$$\varrho := \mathrm{e}^{\gamma}[\cosh(\kappa) + \sinh(\kappa)] = \mathrm{e}^{\gamma+\kappa} = \mathrm{e}^{s},$$

$$\delta := \mathrm{e}^{\gamma}[\cosh(\kappa) - \sinh(\kappa)] = \mathrm{e}^{\gamma-\kappa},$$

we have

$$\boxed{\boldsymbol{X}_k = \frac{1}{2} \begin{bmatrix} \varrho + \delta & \varrho - \delta \\ \varrho - \delta & \varrho + \delta \end{bmatrix} \boldsymbol{X}_{k-1} + \boldsymbol{M}_k + \overline{\boldsymbol{\beta}}, \quad k \in \mathbb{N}.}$$

CLSE of $(\varrho, \delta, \overline{\boldsymbol{\beta}})$ based on observations $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ can be obtained by minimization of

$$\sum_{k=1}^{n} \left\| \boldsymbol{X}_k - \frac{1}{2} \begin{bmatrix} \varrho + \delta & \varrho - \delta \\ \varrho - \delta & \varrho + \delta \end{bmatrix} \boldsymbol{X}_{k-1} - \overline{\boldsymbol{\beta}} \right\|^2$$

with respect to $(\varrho, \delta, \overline{\boldsymbol{\beta}})$ over $\mathbb{R}^4$, and it has the form

$$\widehat{\varrho}_n = \frac{n \sum_{k=1}^{n} \langle \boldsymbol{u}, \boldsymbol{X}_k \rangle \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle - \sum_{k=1}^{n} \langle \boldsymbol{u}, \boldsymbol{X}_k \rangle \sum_{k=1}^{n} \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle}{n \sum_{k=1}^{n} \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle^2 - \left(\sum_{k=1}^{n} \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle \right)^2},$$

$$\widehat{\delta}_n = \frac{n \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{X}_k \rangle \langle \boldsymbol{v}, \boldsymbol{X}_{k-1} \rangle - \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{X}_k \rangle \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{X}_{k-1} \rangle}{n \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{X}_{k-1} \rangle^2 - \left(\sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{X}_{k-1} \rangle \right)^2},$$

$$\widehat{\overline{\boldsymbol{\beta}}}_n = \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{X}_k - \frac{1}{2n} \sum_{k=1}^{n} \begin{bmatrix} \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle & \langle \boldsymbol{v}, \boldsymbol{X}_{k-1} \rangle \\ \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle & -\langle \boldsymbol{v}, \boldsymbol{X}_{k-1} \rangle \end{bmatrix} \begin{bmatrix} \widehat{\varrho}_n \\ \widehat{\delta}_n \end{bmatrix}$$

provided that $n \sum_{k=1}^{n} \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle^2 - \left(\sum_{k=1}^{n} \langle \boldsymbol{u}, \boldsymbol{X}_{k-1} \rangle \right)^2 > 0$ and $n \sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{X}_{k-1}(\omega) \rangle^2 - \left(\sum_{k=1}^{n} \langle \boldsymbol{v}, \boldsymbol{X}_{k-1} \rangle \right)^2 > 0.$

We have

$$(\varrho, \delta, \overline{\boldsymbol{\beta}}) = \left(\mathrm{e}^{\gamma+\kappa}, \mathrm{e}^{\gamma-\kappa}, \left(\int_0^1 \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \, \mathrm{d}s\right) \widetilde{\boldsymbol{\beta}}\right), \qquad (\gamma, \kappa, \widetilde{\boldsymbol{\beta}}) \in \mathbb{R}^4,$$

hence the CLSE of γ, κ and $\widetilde{\boldsymbol{\beta}}$:

$$(\widehat{\gamma}_n, \widehat{\kappa}_n, \widehat{\widetilde{\boldsymbol{\beta}}}_n) = \left(\frac{1}{2}\log(\widehat{\varrho}_n \widehat{\delta}_n), \frac{1}{2}\log\left(\frac{\widehat{\varrho}_n}{\widehat{\delta}_n}\right), \left(\int_0^1 \mathrm{e}^{s\widehat{\widetilde{\boldsymbol{B}}}_n} \, \mathrm{d}s\right)^{-1} \widehat{\overline{\boldsymbol{\beta}}}_n\right), \qquad n \in \mathbb{N},$$

provided that $\widehat{\varrho}_n > 0$ and $\widehat{\delta}_n > 0$, where

$$\left(\int_0^1 \mathrm{e}^{s\widehat{\widetilde{\boldsymbol{B}}}} \, \mathrm{d}s\right)^{-1} = \frac{1}{2\int_0^1 (\widehat{\varrho})^s \, \mathrm{d}s} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2\int_0^1 (\widehat{\delta})^s \, \mathrm{d}s} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

In a similar way, we have $\varrho = \mathrm{e}^s$, hence the CLSE of s is

$$\widehat{s}_n := \log(\widehat{\varrho}_n)$$

provided that $\widehat{\varrho}_n > 0$.

Asymptotics of CLSE : $s = \gamma + \kappa = 0$ and $\kappa > 0$, irreducible and critical (Barczy, Körmendi and P, 2015 JMVA)

If $\boldsymbol{X}_0 = 0$, $\int_{\mathcal{U}_2} \|\boldsymbol{z}\|^8 \, \mu_1(\mathrm{d}\boldsymbol{z}) < \infty$, $\int_{\mathcal{U}_2} \|\boldsymbol{z}\|^8 \, \mu_1(\mathrm{d}\boldsymbol{z}) < \infty$, $\int_{\mathcal{U}_2} \|\boldsymbol{z}\|^8 \, \nu(\mathrm{d}\boldsymbol{z}) < \infty$ and $\widetilde{\beta} \neq \boldsymbol{0}$, then

$$n(\widehat{s}_n - s) \xrightarrow{\mathcal{D}} \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}(\mathcal{M}_{t,1} + \mathcal{M}_{t,2}) - (\mathcal{M}_{1,1} + \mathcal{M}_{1,2}) \int_0^1 \mathcal{Y}_t \, \mathrm{d}t}{\int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t - \left(\int_0^1 \mathcal{Y}_t \, \mathrm{d}t\right)^2} =: \mathcal{I},$$

where $(\boldsymbol{\mathcal{M}}_t)_{t \in \mathbb{R}_+} = (\mathcal{M}_{t,1}, \mathcal{M}_{t,2})_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\mathrm{d}\boldsymbol{\mathcal{M}}_t = ((\mathcal{M}_{t,1} + \mathcal{M}_{t,2} + (\widetilde{\beta}_1 + \widetilde{\beta}_2)t)^+)^{1/2} \, \widetilde{\boldsymbol{C}}^{1/2} \, \mathrm{d}\boldsymbol{\mathcal{W}}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\boldsymbol{\mathcal{M}}_0 = \boldsymbol{0}$, where $(\boldsymbol{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$ is a 2-dimensional standard Wiener process, and

$$\mathcal{Y}_t := \mathcal{M}_{t,1} + \mathcal{M}_{t,2} + (\widetilde{\beta}_1 + \widetilde{\beta}_2)t, \qquad t \in \mathbb{R}_+.$$

If $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$, then

$$n^{3/2}(\widehat{s}_n - s) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{12}{(\widetilde{\beta}_1 + \widetilde{\beta}_2)^2} \int_{\mathcal{U}_2} (z_1 + z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z})\right).$$

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) > 0$, then

$$\begin{bmatrix} n^{1/2}(\widehat{\gamma}_n - \gamma) \\ n^{1/2}(\widehat{\kappa}_n - \kappa) \\ \widehat{\widetilde{\boldsymbol{\beta}}}_n - \widetilde{\boldsymbol{\beta}} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \frac{1}{2}\sqrt{\mathrm{e}^{2(\kappa-\gamma)} - 1} \, \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}\widetilde{\mathcal{W}}_t}{\int_0^1 \mathcal{Y}_t \, \mathrm{d}t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \frac{1}{2}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{\kappa-\gamma}{1-\mathrm{e}^{\gamma-\kappa}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \boldsymbol{\mathcal{M}}_1 - \frac{1}{2}\mathcal{I} \int_0^1 \mathcal{Y}_t \, \mathrm{d}t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

as $n \to \infty$, where $(\widetilde{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, independent from $(\boldsymbol{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$.

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) = 0$ and $\int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) > 0$, then

$$\begin{bmatrix} n^{1/2}(\widehat{\gamma}_n - \gamma) \\ n^{1/2}(\widehat{\kappa}_n - \kappa) \\ \widehat{\widetilde{\boldsymbol{\beta}}}_n - \widetilde{\boldsymbol{\beta}} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \frac{1}{2}\sqrt{\mathrm{e}^{2(\kappa-\gamma)} - 1} \, \widetilde{\mathcal{W}}_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \frac{1}{2}\left(\mathcal{M}_{1,1} + \mathcal{M}_{1,2} - \mathcal{I} \int_0^1 \mathcal{Y}_t \, \mathrm{d}t\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix},$$

where $\widetilde{\mathcal{W}}_1$ is a random variable with standard normal distribution, independent from $(\boldsymbol{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$.

If $\boldsymbol{c} = \boldsymbol{0}$, $\boldsymbol{\mu} = \boldsymbol{0}$ and $\int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) > 0$, then

$$\begin{bmatrix} n^{1/2}(\widehat{\gamma}_n - \gamma) \\ n^{1/2}(\widehat{\kappa}_n - \kappa) \\ n^{1/2}(\widehat{\widetilde{\boldsymbol{\beta}}}_n - \widetilde{\boldsymbol{\beta}}) \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_4 \left(\boldsymbol{0}, \begin{bmatrix} \boldsymbol{R}_{1,1} & \boldsymbol{R}_{1,2} \\ \boldsymbol{R}_{2,1} & \boldsymbol{R}_{2,2} \end{bmatrix} \right) \qquad \text{as } n \to \infty,$$

with

$$\boldsymbol{R}_{1,1} := \frac{\mathrm{e}^{2(\kappa-\gamma)} - 1}{4} \boldsymbol{D}, \qquad \boldsymbol{R}_{2,1} := -\frac{(\widetilde{\beta}_1 - \widetilde{\beta}_2)(\mathrm{e}^{2(\kappa-\gamma)} - 1)}{4(\kappa - \gamma)} \boldsymbol{D},$$

$$\boldsymbol{R}_{2,2} := \int_{\mathcal{U}_2} (z_1 + z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} \int_{\mathcal{U}_2} (z_1^2 - z_2^2) \, \nu(\mathrm{d}\boldsymbol{z}) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$+ \frac{(1 - \mathrm{e}^{2(\gamma-\kappa)})}{4} \left\{ \frac{\kappa - \gamma}{2(1 - \mathrm{e}^{\gamma-\kappa})^2} \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) + \frac{(\widetilde{\beta}_1 - \widetilde{\beta}_2)^2}{(\kappa - \gamma)^2 \mathrm{e}^{2(\gamma-\kappa)}} \right\} \boldsymbol{D},$$

and $\boldsymbol{R}_{1,2} := \boldsymbol{R}_{2,1}^\top$, where $\boldsymbol{D} := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Sketch of proof

This result follows from the following theorem by the continuous mapping theorem and by Slutsky's lemma.

Under the above assumptions,

$$n(\widehat{\varrho}_n - \varrho) \xrightarrow{\mathcal{D}} \mathcal{I}.$$

If $\boldsymbol{c} = \mathbf{0}$ and $\boldsymbol{\mu} = \mathbf{0}$, then

$$n^{3/2}(\widehat{\varrho}_n - \varrho) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{12}{(\widetilde{\beta}_1 + \widetilde{\beta}_2)^2} \int_{\mathcal{U}_2} (z_1 + z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z})\right).$$

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) > 0$, then

$$\begin{bmatrix} n(\widehat{\varrho}_n - \varrho) \\ n^{1/2}(\widehat{\delta}_n - \delta) \\ \widehat{\overline{\boldsymbol{\beta}}}_n - \overline{\boldsymbol{\beta}} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \mathcal{I} \\ \sqrt{1-\delta^2} \, \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}\widetilde{\mathcal{W}}_t}{\int_0^1 \mathcal{Y}_t \, \mathrm{d}t} \\ \boldsymbol{\mathcal{M}}_1 - \frac{1}{2} \mathcal{I} \int_0^1 \mathcal{Y}_t \, \mathrm{d}t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix},$$

where $(\widetilde{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, independent from $(\boldsymbol{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$.

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) = 0$ and $\int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) > 0$, then

$$\begin{bmatrix} n(\widehat{\varrho}_n - \varrho) \\ n^{1/2}(\widehat{\delta}_n - \delta) \\ \widehat{\overline{\boldsymbol{\beta}}}_n - \overline{\boldsymbol{\beta}} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \mathcal{I} \\ \sqrt{1-\delta^2} \, \widetilde{\mathcal{W}}_1 \\ \frac{1}{2}\left(\mathcal{M}_{1,1} + \mathcal{M}_{1,2} - \mathcal{I} \int_0^1 \mathcal{Y}_t \, \mathrm{d}t\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}.$$

If $\boldsymbol{c} = \boldsymbol{0}$, $\boldsymbol{\mu} = \boldsymbol{0}$ and $\int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) > 0$, then

$$\begin{bmatrix} n^{3/2}(\widehat{\varrho}_n - \varrho) \\ n^{1/2}(\widehat{\delta}_n - \delta) \\ n^{1/2}(\widehat{\overline{\boldsymbol{\beta}}}_n - \overline{\boldsymbol{\beta}}) \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_4\left(\boldsymbol{0}, \boldsymbol{S}\right),$$

with

$$\begin{aligned} \boldsymbol{S} := &\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \int_0^1 \int_{\mathcal{U}_2} (\mathrm{e}^{t\widetilde{\boldsymbol{B}}}\boldsymbol{z})(\mathrm{e}^{t\widetilde{\boldsymbol{B}}}\boldsymbol{z})^\top \nu(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}t \end{bmatrix} \\ &+ \frac{3}{4} \int_{\mathcal{U}_2} (z_1 + z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) \begin{bmatrix} \frac{4}{\widetilde{\beta}_1 + \widetilde{\beta}_2} \boldsymbol{e}_1 \\ -\boldsymbol{u} \end{bmatrix} \begin{bmatrix} \frac{4}{\widetilde{\beta}_1 + \widetilde{\beta}_2} \boldsymbol{e}_1 \\ -\boldsymbol{u} \end{bmatrix}^\top \\ &+ (1 - \delta^2) \begin{bmatrix} \boldsymbol{e}_2 \\ -\frac{\widetilde{\beta}_1 - \widetilde{\beta}_2}{2\log(\delta^{-1})} \boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_2 \\ -\frac{\widetilde{\beta}_1 - \widetilde{\beta}_2}{2\log(\delta^{-1})} \boldsymbol{v} \end{bmatrix}^\top . \end{aligned}$$

Decomposition of $\boldsymbol{X}_k$

We have $\boldsymbol{X}_k = U_k \widetilde{\boldsymbol{u}} + V_k \widetilde{\boldsymbol{v}}$ with $\widetilde{\boldsymbol{v}} := \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and

$$U_k := \langle \boldsymbol{u}, \boldsymbol{X}_k \rangle, \qquad V_k := \langle \boldsymbol{v}, \boldsymbol{X}_k \rangle, \qquad k \in \mathbb{Z}_+.$$

Observe $U_k \in \mathbb{R}_+$ and

$$U_k = \varrho U_{k-1} + \widetilde{\varrho}\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}} \rangle + \langle \boldsymbol{u}, \boldsymbol{M}_k \rangle, \qquad V_k = \delta V_{k-1} + \widetilde{\delta}\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle + \langle \boldsymbol{v}, \boldsymbol{M}_k \rangle,$$

for all $k \in \mathbb{Z}_+$ with $\widetilde{\varrho} := \int_0^1 \varrho^s \, \mathrm{d}s$ and $\widetilde{\delta} := \int_0^1 \delta^s \, \mathrm{d}s$, and

$$\widehat{\varrho}_n - \varrho = \frac{n \sum_{k=1}^n \langle \boldsymbol{u}, \boldsymbol{M}_k \rangle U_{k-1} - \sum_{k=1}^n \langle \boldsymbol{u}, \boldsymbol{M}_k \rangle \sum_{k=1}^n U_{k-1}}{n \sum_{k=1}^n U_{k-1}^2 - \left(\sum_{k=1}^n U_{k-1}\right)^2},$$

$$\widehat{\delta}_n - \delta = \frac{n \sum_{k=1}^n \langle \boldsymbol{v}, \boldsymbol{M}_k \rangle V_{k-1} - \sum_{k=1}^n \langle \boldsymbol{v}, \boldsymbol{M}_k \rangle \sum_{k=1}^n V_{k-1}}{n \sum_{k=1}^n V_{k-1}^2 - \left(\sum_{k=1}^n V_{k-1}\right)^2},$$

$$\widehat{\overline{\boldsymbol{\beta}}}_n - \overline{\boldsymbol{\beta}} = \frac{1}{n} \sum_{k=1}^n \boldsymbol{M}_k - \frac{1}{2n} \sum_{k=1}^n \begin{bmatrix} U_{k-1} & V_{k-1} \\ U_{k-1}\rangle & -V_{k-1} \end{bmatrix} \begin{bmatrix} \widehat{\varrho}_n - \varrho \\ \widehat{\delta}_n - \delta \end{bmatrix}.$$

The above theorem follows from the following convergences by the continuous mapping theorem and by Slutsky's lemma.

Under the above assumptions, we have

$$n^{-3/2}\sum_{k=1}^{n} V_{k-1} \xrightarrow{\mathbb{P}} 0 \qquad \text{as} \ \ n\to\infty,$$

$$\sum_{k=1}^{n}\begin{bmatrix} n^{-2}U_{k-1} \\ n^{-3}U_{k-1}^2 \\ n^{-2}V_{k-1}^2 \\ n^{-1}\boldsymbol{M}_k \\ n^{-2}\langle \boldsymbol{u}, \boldsymbol{M}_k\rangle U_{k-1} \\ n^{-3/2}\langle \boldsymbol{v}, \boldsymbol{M}_k\rangle V_{k-1}\end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \int_0^1 \mathcal{Y}_t\,\mathrm{d}t \\ \int_0^1 \mathcal{Y}_t^2\,\mathrm{d}t \\ (1-\delta^2)^{-1}\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}, \boldsymbol{v}\rangle \int_0^1 \mathcal{Y}_t\,\mathrm{d}t \\ \boldsymbol{\mathcal{M}}_1 \\ \int_0^1 \mathcal{Y}_t\,\mathrm{d}\langle \boldsymbol{u}, \boldsymbol{\mathcal{M}}_t\rangle \\ (1-\delta^2)^{-1/2}\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}, \boldsymbol{v}\rangle \int_0^1 \mathcal{Y}_t\,\mathrm{d}\widetilde{\mathcal{W}}_t\end{bmatrix}.$$

In case of $\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}, \boldsymbol{v}\rangle = 0$ the third and sixth coordinates of the limit vector in the second convergence is 0, thus other scaling factors should be chosen for these coordinates, described in the following theorem.

If $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}, \boldsymbol{v} \rangle = 0$, then

$$n^{-1} \sum_{k=1}^{n} V_{k-1} \xrightarrow{\mathbb{P}} \frac{\widetilde{\delta} \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{1-\delta} \qquad \text{as} \quad n \to \infty,$$

$$n^{-1} \sum_{k=1}^{n} V_{k-1}^2 \xrightarrow{\mathbb{P}} \frac{\langle \boldsymbol{V}_0 \boldsymbol{v}, \boldsymbol{v} \rangle}{1-\delta^2} + \frac{(\widetilde{\delta})^2 \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle^2}{(1-\delta)^2} =: M,$$

$$\sum_{k=1}^{n} \begin{bmatrix} n^{-2} U_{k-1} \\ n^{-3} U_{k-1}^2 \\ n^{-1} \langle \boldsymbol{u}, \boldsymbol{M}_k \rangle \\ n^{-2} \langle \boldsymbol{u}, \boldsymbol{M}_k \rangle U_{k-1} \\ n^{-1/2} \langle \boldsymbol{v}, \boldsymbol{M}_k \rangle \\ n^{-1/2} \langle \boldsymbol{v}, \boldsymbol{M}_k \rangle V_{k-1} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \int_0^1 \mathcal{Y}_t \, \mathrm{d}t \\ \int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t \\ \langle \boldsymbol{u}, \boldsymbol{\mathcal{M}}_1 \rangle \\ \int_0^1 \mathcal{Y}_t \, \mathrm{d} \langle \boldsymbol{u}, \boldsymbol{\mathcal{M}}_t \rangle \\ \langle \boldsymbol{V}_0 \boldsymbol{v}, \boldsymbol{v} \rangle^{1/2} \begin{bmatrix} 1 & \frac{\widetilde{\delta} \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{1-\delta} \\ \frac{\widetilde{\delta} \langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}} \rangle}{1-\delta} & M \end{bmatrix}^{1/2} \widetilde{\boldsymbol{\mathcal{W}}}_1 \end{bmatrix},$$

where $\widetilde{\boldsymbol{\mathcal{W}}}_1$ is a 2-dimensional random vector with standard normal distribution, independent from $(\boldsymbol{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$, and

$$\boldsymbol{V}_0 := \operatorname{Var}(\boldsymbol{X}_1 \mid \boldsymbol{X}_0 = \boldsymbol{0}).$$

If $\langle \widetilde{\boldsymbol{C}}\boldsymbol{u}, \boldsymbol{u}\rangle = 0$, then

$$n^{-2}\sum_{k=1}^{n} U_{k-1} \xrightarrow{\mathbb{P}} \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle}{2}, \qquad n^{-3}\sum_{k=1}^{n} U_{k-1}^2 \xrightarrow{\mathbb{P}} \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle^2}{3},$$

$$\sum_{k=1}^{n} \begin{bmatrix} n^{-1/2}\langle \boldsymbol{u}, \boldsymbol{M}_k\rangle \\ n^{-3/2}\langle \boldsymbol{u}, \boldsymbol{M}_k\rangle U_{k-1} \\ n^{-1/2}\langle \boldsymbol{v}, \boldsymbol{M}_k\rangle \\ n^{-1/2}\langle \boldsymbol{v}, \boldsymbol{M}_k\rangle V_{k-1} \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_4(\boldsymbol{0}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\Sigma} := \begin{bmatrix} \boldsymbol{u}^\top \boldsymbol{V}_0^{1/2} \\ \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle}{2}\boldsymbol{u}^\top \boldsymbol{V}_0^{1/2} \\ \boldsymbol{v}^\top \boldsymbol{V}_0^{1/2} \\ \frac{\widetilde{\delta}\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}}\rangle}{1-\delta}\boldsymbol{v}^\top \boldsymbol{V}_0^{1/2} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^\top \boldsymbol{V}_0^{1/2} \\ \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle}{2}\boldsymbol{u}^\top \boldsymbol{V}_0^{1/2} \\ \boldsymbol{v}^\top \boldsymbol{V}_0^{1/2} \\ \frac{\widetilde{\delta}\langle \boldsymbol{v}, \widetilde{\boldsymbol{\beta}}\rangle}{1-\delta}\boldsymbol{v}^\top \boldsymbol{V}_0^{1/2} \end{bmatrix}^\top + \frac{\langle \boldsymbol{u}, \widetilde{\boldsymbol{\beta}}\rangle^2 \boldsymbol{u}^\top \boldsymbol{V}_0 \boldsymbol{u}}{12} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top$$

$$+ \frac{(\boldsymbol{v}^\top \boldsymbol{V}_0 \boldsymbol{v})^2}{1-\delta^2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top .$$

The second convergence follows from

$$\left(\sum_{k=1}^{\lfloor nt\rfloor} \boldsymbol{Z}_k^{(n)}\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{\mathcal{Z}}_t)_{t\in\mathbb{R}_+} \qquad \text{with} \qquad \boldsymbol{Z}_k^{(n)} := \begin{bmatrix} n^{-1}\boldsymbol{M}_k \\ n^{-2}\boldsymbol{M}_k U_{k-1} \\ n^{-3/2}\boldsymbol{M}_k V_{k-1} \end{bmatrix},$$

where the process $(\boldsymbol{\mathcal{Z}}_t)_{t\in\mathbb{R}_+}$ with values in $(\mathbb{R}^2)^3$ is the unique strong solution of the SDE

$$\mathrm{d}\boldsymbol{\mathcal{Z}}_t = \gamma(t, \boldsymbol{\mathcal{Z}}_t) \begin{bmatrix} \mathrm{d}\boldsymbol{\mathcal{W}}_t \\ \mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t \end{bmatrix}, \quad t \in \mathbb{R}_+, \qquad \boldsymbol{\mathcal{Z}}_0 = \boldsymbol{0},$$

where $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ and $(\widetilde{\boldsymbol{\mathcal{W}}}_t)_{t\in\mathbb{R}_+}$ are independent 2-dimensional standard Wiener processes, and

$$\gamma(t, \boldsymbol{x}) := \begin{bmatrix} (\langle \boldsymbol{u}, \boldsymbol{x}_1 + t\boldsymbol{\beta}\rangle^+)^{1/2}\widetilde{\boldsymbol{C}}^{1/2} & 0 \\ (\langle \boldsymbol{u}, \boldsymbol{x}_1 + t\boldsymbol{\beta}\rangle^+)^{3/2}\widetilde{\boldsymbol{C}}^{1/2} & 0 \\ 0 & \frac{\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}, \boldsymbol{v}\rangle^{1/2}}{(1-\delta^2)^{1/2}} \langle \boldsymbol{u}, \boldsymbol{x}_1 + t\boldsymbol{\beta}\rangle \widetilde{\boldsymbol{C}}^{1/2} \end{bmatrix}$$

for $t \in \mathbb{R}_+$ and $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \in (\mathbb{R}^2)^3$.

From a semimartingale limit theorem of Jacod and Shiryaev (2003) :

A martingale limit theorem (Ispány and P, 2010 AMH)

Let $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ be a continuous function. Assume that uniqueness in the sense of probability law holds for the SDE

$$\mathrm{d}\mathcal{Z}_t = \gamma(t, \mathcal{Z}_t)\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\mathcal{Z}_0 = \boldsymbol{z}_0$ for all $\boldsymbol{z}_0 \in \mathbb{R}^d$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is an r-dimensional standard Wiener process. Let $(\mathcal{Z}_t)_{t\in\mathbb{R}_+}$ be a solution with initial value $\mathcal{Z}_0 = \boldsymbol{0}$. For each $n \in \mathbb{N}$, let $(\boldsymbol{Z}_k^{(n)})_{k\in\mathbb{N}}$ be a sequence of martingale differences in $\mathbb{R}^d$ with respect to a filtration $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$. Let $\mathcal{Z}_t^{(n)} := \sum_{k=1}^{\lfloor nt\rfloor} \boldsymbol{Z}_k^{(n)}$ for $t \in \mathbb{R}_+$. Suppose

1. $$\sup_{t\in[0,T]} \left\| \sum_{k=1}^{\lfloor nt\rfloor} \operatorname{Var}(\boldsymbol{Z}_k^{(n)} \mid \mathcal{F}_{k-1}^{(n)}) - \int_0^t \gamma(s, \mathcal{Z}_s^{(n)})\gamma(s, \mathcal{Z}_s^{(n)})^\top \mathrm{d}s \right\| \xrightarrow{\mathbb{P}} 0,$$

2. $$\sum_{k=1}^{\lfloor nT\rfloor} \mathbb{E}\big(\|\boldsymbol{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{Z}_k^{(n)}\|>\theta\}} \mid \mathcal{F}_{k-1}^{(n)}\big) \xrightarrow{\mathbb{P}} 0 \text{ for all } \theta > 0.$$

for all $T > 0$, Then $(\mathcal{Z}_t^{(n)})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\mathcal{Z}_t)_{t\in\mathbb{R}_+}$.

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