A Branching Process with Immigration in Varying Environments

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This is a joint work with Edward Omey from Faculty of Economics and Business, KU Leuven, Brussels, Belgium.

It presents the results from our paper:

"A Branching Process with Immigration in Varying Environments"

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It is known that a branching process is said to be in **varying environments** if the offspring distributions of the particles change with time.

Most of the classical branching processes have been studied also in varying environments.

A few papers on this topic are:

- P. Jagers, (1974), J. Appl. Prob., 11, 174–178.
- A. Agresti, (1975), J. Appl. Prob., 12, 39-46.
- N. Keiding and J. Nielsen, (1975), J. Appl. Prob., 12, 135-141.
- T. Fujimagari, (1980), Adv. Appl. Prob., 12, 350-366.
- H. Cohn and P. Jagers, (1994), Annals of Appl. Prob., 4(1), 184-193.
- J. D'Souza, (1994), Adv. Appl. Prob., 26, 698-714.
- K. Mitov, A. Pakes and G. Yanev, (2003), Stat. & Probab. Lett., 65, 379-388.
- S.D. Batra and S.C. Gupta, (2005), J. Inti. Soc. Agril. Statist., 59(3), 233–236.

The problem we consider

In the talk we consider Bienaymé-Galton-Watson branching processes with **geometric distribution** of the offspring and an **immigration component.**

The parameter of the offspring distribution changes from generation to generation so that:

- the mean number of the offspring is always equal to one *(critical process)*

- the variance tends to infinity.

The investigation of such processes was initiated in

K.M. and G.Yanev, A critical branching process with increasing offspring variance, Proc. 30-th Spring Conf. of UBM, Sofia, BAS, 2002, 166-171,

where we consider the processes without immigration.

Motivation - 1

- It is known that the limiting behaviour of critical branching processes essentially defers in cases with finite and infinite variance.
- The immigration changes the behaviour of any branching process, in dependence of the number of immigrants. We consider two regimes of immigration: with finite and infinite mean number of immigrants.
- Both varying environments and the immigration make the BGW branching processes more flexible and enlarge the possible applications of these processes.

Motivation - 2

Our work was motivated also by the pure technical problem for the asymptotic behaviour of the sum of the following kind:

$$\sum_{j=1}^{n} \frac{1}{1 + \sum_{k=j}^{n} c(k)},$$

depending on the behaviour of $c(n)$, as $n \to \infty$.

Definitions and notations

Definition

$$Y_0 = 0, \quad Y_{n+1} = \sum_{i=1}^{Y_n+I_n} X_i(n+1), (\sum_{i=1}^{0} . = 0), \quad n = 0, 1, 2, \ldots.$$

The random variables $X_i(n), i = 1, 2, \ldots; n = 0, 1, 2, \ldots$ are ***independent and identically distributed for every fixed*** n *but the distribution varies with n form generation to generation.*

The numbers of immigrants $\{I_n : \; n = 1, 2, \ldots\}$ are iid integer valued non negative random variables, independent of X's with p.g.f. $g(s) = E[s^{I_n}]$.

Definitions and notations

For the offspring distribution we assume that

$$\begin{aligned}\Pr(X_i(n) = 0) &= 1 - p(n),\\ \Pr(X_i(n) = k) &= p(n)^2(1 - p(n))^{k-1}, k = 1, 2, \dots,\end{aligned} \quad (1)$$

for $i = 1, 2, \dots$ and $n = 1, 2, \dots$.
We denote

$$m(n) = E[X_i(n)] = 1, \sigma_n^2 = Var[X_i(n)] = 2c(n) = \frac{2(1 - p(n))}{p(n)}.$$

Then the pgf $f_n(s) = E[s^{X_i(n)}]$ has the following form

$$f_n(s) = 1 - \frac{1 - s}{1 + c(n)(1 - s)}.$$

Definitions and notations

Denote by

$$F(j,n;s) = f_j(f_{j+1}(f_{j+2}(\ldots(f_n(s))\ldots))).$$

Then we have

$$F(j,n;s) = 1 - \frac{1-s}{1+B(j,n)(1-s)},$$

where $B(j,n) = \displaystyle\sum_{k=j}^{n} c(k)$.

In particular

$$F(n;s) = E[s^{Z_n}|Z_0 = 1] = F(1,n;s) = 1 - \frac{1-s}{1+B(n)(1-s)},$$

where $B(n) = \displaystyle\sum_{k=1}^{n} c(k)$.

Basic equations

Denote the pgf of Y_n by $\Phi(n;s) = E[s^{Y_n}|Y_0 = 0]$. The following equation holds true

$$\Phi(n;s) = \prod_{j=1}^{n} g(F(j,n;s)) = \prod_{j=1}^{n} g\left(1 - \frac{1-s}{1+B(j,n)(1-s)}\right).$$

For $s = 0$ we have

$$\Phi(n;0) = \prod_{j=1}^{n} g(F(j,n;0)) = \prod_{j=1}^{n} g\left(1 - \frac{1}{1+B(j,n)}\right).$$

These equations are the basic tools in the study of the process Y_n.

For the offspring distribution we assume that for $\alpha > 0$

$$p(n) \sim \frac{L(n)}{n^\alpha} \to 0, \quad n \to \infty. \tag{2}$$

Then

$$\sigma_n^2 = 2c(n) \sim \frac{2}{p(n)} \sim n^\alpha L_1(n) \to \infty, n \to \infty. \tag{3}$$

The process is critical but the variance increases to infinity with *n*.

Assumptions for the immigration

For the immigration we assume either:
The first two moments are finite.

$$m_I := g'(1) < \infty, \quad b_I := g''(1) < \infty. \tag{4}$$

or
Infinite mean of the number of immigrants.

$$g(s) = 1 - R(1/(1-s)) \text{ where } R(x) \in RV(-\beta), \tag{5}$$

$R(x)$ is non increasing and β is such that $1/(1+\alpha) < \beta < 1$.

As usual we consider:

- Probability for non-visiting the state zero.
- Asymptotic of the moments.
  when they are finite
- Limiting distributions.

Depending on the immigration component.

Probability for non-visiting zero

Theorem

Assume that

$$m(n) = 1, \quad c(n) \sim n^{\alpha} L_1(n), n \to \infty, \quad \alpha > 0,$$

and the first and second moments of the number of immigrants are finite:

$$m_I := g'(1) < \infty, \quad b_I := g''(1) < \infty.$$

Then

$$\Pr(Y_n > 0) \sim m_I \frac{\log n}{c(n)} \sim n^{-\alpha} L_2(n), \quad n \to \infty.$$

Theorem

Suppose that

$$m(n) = 1, \quad c(n) \sim n^{\alpha} L_1(n), n \to \infty, \quad \alpha > 0,$$

and

$$g(s) = 1 - R(1/(1-s)), \text{ where } R(x) = x^{-\beta} L_1(x).$$

R(x) is non increasing and $1/(1+\alpha) < \beta < 1$*.*
Then

$$\Pr(Y_n > 0) \sim C(\alpha, \beta) n R(nc(n)) \sim C(\alpha, \beta) n^{1-(1+\alpha)\beta} L_3(n),$$

where

$$C(\alpha, \beta) = (1+\alpha)^{\beta-1} B(\frac{1}{1+\alpha}, 1-\beta),$$

and $B(.,.)$ *is Euler's beta function.*

Comment

Under the conditions of the theorem $\alpha > (1 + \alpha)\beta - 1$. Then, in the second case the probability for non-visiting the state zero decreases to zero more slowly.

Theorem

Assume that

$$m(n) = 1, \quad c(n) < \infty \quad n = 1, 2, \ldots,$$

and

$$m_I := g'(1) < \infty, \quad b_I := g''(1) < \infty.$$

Then

$$E[Y_n] = m_I n, \;\; n \geq 1,$$
$$Var[Y_n] = 2m_I \sum_{j=1}^{n} B(j,n) + d_I n, \;\; n \geq 1,$$

where $d_I = b_I + m_I - m_I^2 = Var[I_n]$.

Asymptotic of the moments

Corolary

Under the condition of the Theorem and $c(n) \sim n^{\alpha} L_1(n)$,

$$Var[Y_n] = \frac{2m_l}{\alpha + 2} n^2 c(n)(1 + o(1)), \quad n \to \infty.$$

Theorem

Assume that

$$m(n) = 1, \quad c(n) \sim n^{\alpha} L_1(n), n \to \infty, \quad \alpha > 0,$$

and the first and second moments of the number of immigrants are finite:

$$m_I := g'(1) < \infty, \quad b_I := g''(1) < \infty.$$

$$\lim_{n\to\infty} \Pr\left(\frac{\log Y_n - \log c(n)}{\log n} \leq x \,\middle|\, Y_n > 0 \right) = x, \;\; x \in (0,1).$$

Limit theorems

Theorem

Suppose that

$m(n) = 1, \;\; c(n) \sim n^{\alpha} L_1(n), n \to \infty, \;\; \alpha > 0,$

$g(s) = 1 - R(1/(1-s)),$

where $R(x) = x^{-\beta} L_1(x), \;\; R(x)$ *is non increasing,*

and $1/(1+\alpha) < \beta < 1$. *Then for* $\lambda > 0$*,*

$$\lim_{n\to\infty} \frac{1 - \Phi(n, e^{-\lambda/nc(n)})}{nR(nc(n))} = \int_0^1 \left(\frac{1}{\lambda} + \frac{1}{\alpha+1}(1 - x^{\alpha+1}) \right)^{-\beta} dx.$$

By the continuity theorem for Laplace transforms we have:

Corolary

Under the conditions of Theorem we have for every $x > 0$,

$$\lim_{n\to\infty} \Pr\left(\frac{Y_n}{nc(n)} \le x | Y_n > 0\right) = D(x;\alpha,\beta),$$

where $D(x;\alpha,\beta)$ has Laplace transform

$$\hat{D}(\lambda;\alpha,\beta) = 1 - \frac{1}{C(\alpha,\beta)} \int_0^1 \left(\frac{1}{\lambda} + \frac{1}{\alpha+1}(1 - x^{\alpha+1})\right)^{-\beta} dx,$$

and

$$C(\alpha,\beta) = (1+\alpha)^{\beta-1} B(\frac{1}{1+\alpha}, 1-\beta),$$

Tail behaviour of the limit distribution

The distribution function $D(x; \alpha, \beta)$ is not explicitly expressed. What can we say about it?

It is not difficult to see that

$$\frac{1 - \hat{D}(\lambda; \alpha, \beta)}{\lambda} \sim \frac{\lambda^{\beta-1}}{C(\alpha, \beta)}, \quad \lambda \downarrow 0.$$

Then by a Tauberian theorem for Laplace transform we have

$$1 - D(x; \alpha, \beta) \sim \frac{x^{-\beta}}{C(\alpha, \beta)\Gamma(\beta)}, \quad x \to \infty.$$

So, the limiting distribution has heavy tail with exponent $\beta < 1$.

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