Statistical inference for critical 2-type Galton–Watson processes with immigration

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The model

Definition

The process $(X_n)_{n\in\mathbb{N}}$ is a 2-type Galton–Watson process with immigration if

$$X_n = \sum_{k=1}^{X_{n-1,1}} \xi_{n,k,1} + \sum_{k=1}^{X_{n-1,2}} \xi_{n,k,2} + \varepsilon_n \, ,$$

where

- $(\xi_{n,k,i})_{n,k\in\mathbb{N}}$ consists of i.i.d. nonnegative, integer valued random vectors for $i = 1, 2$, namely the offspring vectors of a type i individual;
- $(\varepsilon_n)_{n\in\mathbb{N}}$ are i.i.d. nonnegative, integer valued random vectors, namely the immigration vectors;
- the offspring and immigration vectors are independent from each other.

For the sake of simplicity, we suppose $X_0 = 0$.

Parameters

Let

$$m_\xi := \begin{bmatrix} \mathbb{E}(\xi_{1,1,1}) & \mathbb{E}(\xi_{1,1,2}) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \qquad m_\varepsilon := \mathbb{E}(\varepsilon_1),$$

denote the offspring mean matrix and the immigration mean vector, and

$$C_i := \mathrm{Var}(\xi_{1,1,i}), \qquad C_\varepsilon := \mathrm{Var}(\varepsilon_1)$$

denote the offspring variances and the immigration variance.

We distinguish 3 cases based on ϱ, the spectral radius of m_ξ.

- If $\varrho < 1$, then the process is subcritical.
- If $\varrho = 1$, then the process is critical.
- If $\varrho > 1$, then the process is supercritical.

If we assume criticality, the the eigenvalues of m_ξ are 1 and $\alpha + \delta - 1$.

Eigenvectors of m_ξ

Let

$$u_{\text{left}} := \frac{1}{2-\alpha-\delta}\begin{bmatrix}\gamma+1-\delta\\ \beta+1-\alpha\end{bmatrix}, \quad u_{\text{right}} := \frac{1}{\beta+1-\alpha}\begin{bmatrix}\beta\\ 1-\alpha\end{bmatrix}$$

and

$$v_{\text{left}} := \frac{1}{\beta+1-\alpha}\begin{bmatrix}-(1-\alpha)\\ \beta\end{bmatrix}, \quad v_{\text{right}} := \frac{1}{2-\alpha-\delta}\begin{bmatrix}-(\beta+1-\alpha)\\ \gamma+1-\delta\end{bmatrix}$$

denote left and right eigenvectors of the eigenvalues 1 and $\alpha+\delta-1$ respectively.
Then the powers of the matrix m_ξ behaves the following way

$$m_\xi^k = u_{\text{right}} u_{\text{left}}^\top + (\alpha+\delta-1)^k v_{\text{right}} v_{\text{left}}^\top, \quad k \in \mathbb{N}.$$

A limit theorem for the process

Let

$$C_\xi := \frac{\beta C_1 + (1-\alpha) C_2}{\beta + 1 - \alpha}.$$

Theorem (Ispány & Pap 2012)

Let $(X_n)_{n\in\mathbb{N}}$ be a critical, 2-type Galton-Watson process with immigration. Suppose $X_0 = 0$, $m_\varepsilon \neq 0$, and $\mathbb{E}(\|\xi_{1,1,i}\|^2) < \infty$, $\mathbb{E}(\|\varepsilon_1\|^2) < \infty$. Then

$$(n^{-1} X_{\lfloor nt \rfloor})_{t\geq 0} \xrightarrow{\mathcal{D}} (\mathcal{Y}_t u_{\mathrm{right}})_{t\geq 0} \qquad \text{as} \quad n \to \infty,$$

where $\mathcal{Y}$ is the unique strong solution of the SDE

$$d\mathcal{Y}_t = \langle u_{\mathrm{left}}, m_\varepsilon \rangle dt + \sqrt{\langle C_\xi u_{\mathrm{left}}, u_{\mathrm{left}} \rangle \mathcal{Y}_t^+}\, d\mathcal{W}_t, \quad \mathcal{Y}_0 = 0,$$

and $(\mathcal{W}_t)_{t\geq 0}$ is a standard Wiener process.

CLS estimator

Introduce the following martingale differences

$$M_k := X_k - \mathbb{E}(X_k | X_{k-1}) = X_k - m_\xi X_{k-1} - m_\varepsilon.$$

If we minimize the sum

$$\sum_{k=1}^{n} \|M_k\|^2$$

with respect to m_ξ we get the so called conditional least squares estimate

$$\widehat{m}_\xi = \sum_{k=1}^{n} (X_k - m_\varepsilon) X_{k-1}^\top \left(\sum_{k=1}^{n} X_{k-1} X_{k-1}^\top \right)^{-1} = B_n A_n^{-1}.$$

Alternative form for the estimate

By the continous mapping theorem

$$n^{-3}A_n = \int_0^1 n^{-2} X_{\lfloor ns \rfloor} X_{\lfloor ns \rfloor}^\top \, ds \xrightarrow{\mathcal{D}} \int_0^1 \mathcal{Y}_s^2 \, ds \, u_{\mathrm{left}} u_{\mathrm{left}}^\top.$$

Since $u_{\mathrm{left}} u_{\mathrm{left}}^\top$ is singular we have to find another form for handling our estimate.

Expressing the inverse of A_n with it's adjugate matrix before finding the limit helps

$$\widehat{m}_\xi = \frac{B_n \operatorname{adj}(A_n)}{\det(A_n)}$$

Rewriting the process in terms of eigenvectors

Define

$$U_k := \langle u_{\mathrm{left}}, X_k \rangle, \quad V_k := \langle v_{\mathrm{left}}, X_k \rangle.$$

Then

$$\begin{aligned} X_k &= U_k u_{\mathrm{right}} + V_k v_{\mathrm{right}} \\ U_k &= U_{k-1} + \langle u_{\mathrm{left}}, M_k \rangle + \langle u_{\mathrm{left}}, m_\varepsilon \rangle, \\ V_k &= (\alpha + \delta - 1) V_{k-1} + \langle v_{\mathrm{left}}, M_k \rangle + \langle v_{\mathrm{left}}, m_\varepsilon \rangle. \end{aligned}$$

Using the continous mapping theorem yields

$$n^{-(\ell+1)} \sum_{k=1}^{n} U_{k-1}^{\ell} \xrightarrow{\mathcal{D}} \int_0^1 \langle u_{\mathrm{left}}, \mathcal{Y}_t u_{\mathrm{right}} \rangle^{\ell} \, dt = \int_0^1 \mathcal{Y}_t^{\ell} \, dt,$$

$$n^{-(\ell+1)} \sum_{k=1}^{n} V_{k-1}^{\ell} \xrightarrow{\mathcal{D}} \int_0^1 \langle v_{\mathrm{left}}, \mathcal{Y}_t u_{\mathrm{right}} \rangle^{\ell} \, dt = 0.$$

Rewriting $\det(A_n)$ in terms of eigenvectors

Using the variables U_k and V_k we can express $\det(A_n)$ in the following way

$$\det(A_n) = \sum_{k=1}^{n} X_{k-1,1}^2 \sum_{k=1}^{n} X_{k-1,2}^2 - \left(\sum_{k=1}^{n} X_{k-1,1} X_{k-1,2} \right)^2$$
$$= \sum_{k=1}^{n} U_{k-1}^2 \sum_{k=1}^{n} V_{k-1}^2 - \left(\sum_{k=1}^{n} U_{k-1} V_{k-1} \right)^2 .$$

We need to determine the asymptotics of

$$\sum_{k=1}^{n} V_{k-1}^2, \qquad \sum_{k=1}^{n} U_{k-1} V_{k-1} .$$

Tools: a limit theorem

A martingale limit theorem (Ispány & Pap 2009)

Let $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ be continuous and assume that a unique weak solution exists for the SDE

$$d\mathcal{Z}_t = \gamma(t, \mathcal{Z}_t)\, d\mathcal{W}_t, \qquad t \geq 0,$$

with initial value $\mathcal{Z}_0 = z_0$ for all $z_0 \in \mathbb{R}^d$. Let $(\mathcal{Z}_t)_{t\geq 0}$ be a solution with initial value $\mathcal{Z}_0 = 0$. For each $n \in \mathbb{N}$, let $(Z_k^{(n)})_{k\in\mathbb{N}}$ be a sequence of martingale differences in $\mathbb{R}^d$ with respect to a filtration $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$. Let $\mathcal{Z}_t^{(n)} := \sum_{k=1}^{\lfloor nt\rfloor} Z_k^{(n)}$ for $t \geq 0$. Suppose

1. $$\sup_{t\in[0,T]} \left\| \sum_{k=1}^{\lfloor nt\rfloor} \mathbb{E}[Z_k^{(n)}(Z_k^{(n)})^\top \mid \mathcal{F}_{k-1}^{(n)}] - \int_0^t \gamma(s, \mathcal{Z}_s^{(n)})\gamma(s, \mathcal{Z}_s^{(n)})^\top ds \right\| \xrightarrow{P} 0,$$
2. $$\sum_{k=1}^{\lfloor nT\rfloor} \mathbb{E}\left[\|Z_k^{(n)}\|^2 \mathbb{1}_{\{\|Z_k^{(n)}\|>\theta\}} \mid \mathcal{F}_{k-1}^{(n)} \right] \xrightarrow{P} 0 \text{ for all } \theta > 0$$

for all $T > 0$. Then $\mathcal{Z}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{Z}$.

Tools: good bounds

Lemma

Let $(_n)_{n\in\mathbb{N}}$ be a critical 2-type Galton–Watson process with immigration. Suppose $X_0 = 0$, $\mathbb{E}(\|\xi_{1,1,i}\|^\ell) < \infty$ and $\mathbb{E}(\|\varepsilon_1\|^\ell) < \infty$ with some $\ell \in \mathbb{N}$. Then

1. for all $i, j \in \mathbb{Z}_+$ with $\max\{i, j\} \le \lfloor \ell/2 \rfloor$ and $\theta > i + j/2 + 1$
$$n^{-\theta} \sum_{k=1}^{n} |U_k^i V_k^j| \xrightarrow{P} 0,$$
2. for all $i, j \in \mathbb{Z}_+$ with $\max\{i, j\} \le \ell$ and $T > 0$, $\theta > i + j/2 + (i + j)/\ell$
$$n^{-\theta} \sup_{t\in[0,T]} |U_k^i V_k^j| \xrightarrow{P} 0,$$
3. for all $i, j \in \mathbb{Z}_+$ with $\max\{i, j\} \le \lfloor \ell/4 \rfloor$ and $t > 0$, $\theta > i + j/2 + 1/2$
$$n^{-\theta} \sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} U_k^i V_k^j - \mathbb{E}(U_k^i V_k^j \mid \mathcal{F}_{k-2}) \right| \xrightarrow{P} 0.$$

Asymptotics for $\sum V_{k-1}^2$

Our aim is to separate a martingale from $\sum_{k=1}^n V_{k-1}^2$. Note that

$$\mathbb{E}(\langle v_{\mathrm{left}}, M_k\rangle^2 \mid \mathcal{F}_{k-1}) = \langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle U_{k-1} + \mathrm{const} \times V_{k-1} + \mathrm{const}.$$

Taking conditional expectation yields

$$\mathbb{E}\left[V_{k-1}^2 \mid \mathcal{F}_{k-2}\right] = \mathbb{E}\left[((\alpha+\delta-1)V_{k-2} + \langle v_{\mathrm{left}}, M_{k-1} + m_\varepsilon\rangle)^2 \mid \mathcal{F}_{k-2}\right]$$
$$= (\alpha+\delta-1)^2 V_{k-2}^2 + \langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle U_{k-2} + \mathrm{const} \times V_{k-2} + \mathrm{const}.$$

Finally

$$\sum_{k=1}^n V_{k-1}^2 = \frac{\langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle}{1-(\alpha+\delta-1)^2} \sum_{k=1}^n U_{k-1} + \mathrm{martingale} + \mathrm{leftovers}$$

Asymptotics for $\det(A_n)$

By the continuos mapping theorem and martingae separation we have

$$n^{-3}\sum_{k=1}^{n} U_{k-1}^2 \xrightarrow{\mathcal{D}} \int_0^1 \mathcal{Y}_t^2\,dt, \quad n^{-2}\sum_{k=1}^{n} V_{k-1}^2 \xrightarrow{\mathcal{D}} \frac{\langle C_\xi v_{\text{left}}, v_{\text{left}}\rangle}{2-\alpha-\delta}\int_0^1 \mathcal{Y}_t\,dt.$$

Similarly, separating a martingale part from $\sum U_{k-1}V_{k-1}$, we can show that

$$n^{-5/2}\sum_{k=1}^{n} U_{k-1}V_{k-1} \xrightarrow{P} 0.$$

Finally

$$n^{-5}\det(A_n) \xrightarrow{\mathcal{D}} \frac{\langle C_\xi v_{\text{left}}, v_{\text{left}}\rangle}{1-(\alpha+\delta-1)^2}\int_0^1 \mathcal{Y}_t\,dt \int_0^1 \mathcal{Y}_t^2\,dt.$$

A limit theorem for the estimator

Theorem (Körmendi & Pap 2014)

Let $(X_n)_{n\in\mathbb{N}}$ be a critical, 2-type Galton-Watson process with immigration. Suppose $X_0 = 0$, $m_\varepsilon \neq 0$, and $\mathbb{E}(\|\xi_{1,1,i}\|^8) < \infty$, $\mathbb{E}(\|\varepsilon_1\|^8) < \infty$. If $\langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle > 0$, then

$$\sqrt{n}(\widehat{m}_\xi - m_\xi) \xrightarrow{\mathcal{D}} \sqrt{1-(\alpha+\delta-1)^2}\,\frac{\int_0^1 \mathcal{Y}_t C_\xi^{1/2}\, d\widetilde{\mathcal{W}}_t v_{\mathrm{left}}^\top}{\langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle^{1/2} \int_0^1 \mathcal{Y}_t\, dt}$$

as $n \to \infty$, where $\mathcal{Y}$ is the unique strong solution of the SDE

$$d\mathcal{Y}_t = \langle u_{\mathrm{left}}, m_\varepsilon\rangle dt + \sqrt{\langle C_\xi u_{\mathrm{left}}, u_{\mathrm{left}}\rangle \mathcal{Y}_t^+}\, d\mathcal{W}_t, \quad \mathcal{Y}_0 = 0,$$

and $(\widetilde{\mathcal{W}}_t)_{t\geq 0}$ is a 2-dimensional Wiener process independent from $\mathcal{W}$.

A limit theorem for the estimator

Since the estimator for m_ξ is weakly consistent we can define an estimator for the criticality parameter ϱ as well.

Theorem (Körmendi & Pap 2014)

Let $(X_n)_{n\in\mathbb{N}}$ be a critical, 2-type Galton-Watson process with immigration. Suppose $X_0 = 0$, $m_\varepsilon \neq 0$, and $\mathbb{E}(\|\xi_{1,1,i}\|^8) < \infty$, $\mathbb{E}(\|\varepsilon_1\|^8) < \infty$. If $\langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}} \rangle > 0$, then

$$n(\widehat{\varrho} - 1) \xrightarrow{\mathcal{D}} \frac{\int_0^1 \mathcal{Y}_t \, d\left(\mathcal{Y}_t - t\langle u_{\mathrm{left}}, m_\varepsilon \rangle\right)}{\int_0^1 \mathcal{Y}_t^2 \, dt}$$

as $n \to \infty$.

Possible generalizations

1. Continous time branching processes.

   We are working on this, but only have results with heavy restrictions on the structure of the equivalent of m_ξ.

2. Increasing the number of types to general d-type GWI processes.

   If $d > 2$ then we don't have a natural estimator for the criticality parameter ϱ, eigenvalues with multiplicity greater than 1 may cause problems.

Thank you for your attention!