ESCAPE FROM THE BOUNDARY IN MARKOV POPULATION PROCESSES

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Birth-Death process of Bare Bones

Stochasticity in the adaptive dynamics of evolution: the bare bones (2011) Klebaner, Sagitov S., Vatutin V., Haccou P., Jagers P.
A continuous time version of the bare bones model is a birth-death process X_N on $\mathbb{Z}_+^2$,

Carrying capacity enters via death rates.
The transition rates are as follows:

$$
\begin{array}{llll}
X \to X + (1,0) & \text{at rate} & aX_1; \\
X \to X + (-1,0) & \text{at rate} & X_1\{(X_1/N) + \gamma(X_2/N)\}; \\
X \to X + (0,1) & \text{at rate} & bX_2; \\
X \to X + (0,-1) & \text{at rate} & X_2\{\gamma(X_1/N) + (X_2/N)\}.
\end{array}
$$

More general processes

X_N is a Markov population process on $\mathbb{Z}_+^d$ having transition rates

$$X \;\to\; X + J \quad \text{at rate } Ng^J(N^{-1}X), \qquad X \in \mathbb{Z}_+^d,\ J \in \mathcal{J} \subset \mathbb{Z}^d \tag{1}$$

Denote the density process $x_N := N^{-1}X_N$.
Then x_N can be described by the equation

$$x_N(t) \;=\; x_N(0) + \int_0^t F(x_N(u))\,du + m_N(t), \tag{2}$$

where

$$F(x) := \sum_{J\in\mathcal{J}} Jg^J(x), \qquad x \in \mathbf{Re}_+^d, \tag{3}$$

and m_N is a vector valued martingale.

Approximation on finite time intervals

Theorem (Kurtz, 1970)

If $\lim_{N\to\infty} x_N(0) = x_0$,
then $\sup_{0\leq t\leq T} |x_N(t) - x(t)| \to 0$ *in distribution,*
for any finite $T > 0$*, where* x *solves corresponding deterministic equations*

$$\dot{x} = F(x), \quad x(0) = x_0. \tag{4}$$

In the Bare Bones example

$$F(\mathbf{x}) = \begin{pmatrix} x(a - x - \gamma y) \\ y(b - \gamma x - y) \end{pmatrix}. \tag{5}$$

The problem of small initial mutant population

If the number of established population is around its equilibrium a and the number of mutants is fixed (typically 1) then the density process has its initial condition convergent to $(a, 0)$, a fixed point of F.
Therefore by the Kurtz' Theorem $x_N(t) \to (a, 0)$ for all finite t.

Since $(a, 0)$ is an unstable fixed point, its approximation on finite intervals is not informative.

Time of escape

The problem of being near an unstable fixed point is just the same in systems without noise.

Consider $\dot{x} = g(x)$ with 0 as unstable fixed point for g, $g(0) = 0$. Due to continuous dependence on initial conditions, the approximation to the solution with $x(0) = \varepsilon$ as $\varepsilon \to 0$, on any finite time interval is 0, $x(t) \to 0$.

Example:
If $g(x) = x$ then $x(t) = \varepsilon e^t$. Hence if $T_\varepsilon = \ln 1/\varepsilon$, then $x(t + T_\varepsilon) = e^t$, $\dot{y}(t) = y(t)$, $y(0) = 1$.
With change of time $t + T_\varepsilon$, the limiting dynamics is the same but with a different initial condition.
T_ε is sharp, cT_ε does not escape from 0 for $c < 1$, and ends up at the stable fixed point for $c > 1$.

T_ε is the time of escape.
We want to find time of escape for our non linear, stochastic system.

Linear g. Feller diffusion.

$$X(t) = \varepsilon + \int_0^t X(s)ds + \sqrt{\varepsilon}\int_0^t \sqrt{X}(s)dB(s).$$

$X(t) = \varepsilon Z(t)$, where

$$Z(t) = 1 + \int_0^t Z(s)ds + \int_0^t \sqrt{Z}(s)dB(s).$$

$$W = \lim_{t\to\infty} \underbrace{Z(t)e^{-t}}_{X(\ln 1/\varepsilon)}.$$

$$X(\ln 1/\varepsilon + t) = \varepsilon Z(\ln 1/\varepsilon + t) = Z(\ln 1/\varepsilon + t)e^{-t-\ln 1/\varepsilon}e^t \to We^t.$$

$$Y(t) = W + \int_0^t Y(s)ds.$$

Non-linear $g(x) = x(1-x)$.

$$\dot{x}(t) = g(x(t)), \; x(0) = \varepsilon.$$

Lemma

Let $\tau_t^\varepsilon = -c \ln \varepsilon + t$. Then as $\varepsilon \to 0$, the time-changed function $\tilde{x}_t = x(\tau_t^\varepsilon) = x(-c \ln \varepsilon + t)$ converges to $y(t)$

1. *if $c < 1$, $y(t)=0$*
2. *if $c = 1$, y is a solution of the same equation $\dot{y}(t) = g(y)$ with $y_0 = \frac{1}{2}$*
3. *if $c > 1$, $y(t)=1$.*

Non-linear $g(x) = x(1-x)$, stochastic.

$$X(t) = \varepsilon + \int_0^t g(X(s))ds + \sqrt{\varepsilon}\int_0^t \sqrt{X}(s)dB(s).$$

Theorem (nearly proved)

$$X(\ln 1/\varepsilon + t) \Rightarrow Y(t), \quad Y(t) = \frac{W}{1+W} + \int_0^t g(Y(s))ds,$$

where

$$W = \lim_{t\to\infty} Z(t)e^{-t}$$

and $Z(t)$ *has linear drift and driven by the same BM* $B(t)$.

Approximations to deterministic Bare Bones

$x(0) = a,\ y(0) = \varepsilon$

$$\begin{aligned} \dot{x} &= (a - x - \gamma y)x, \\ \dot{y} &= (b - \gamma x - y)y. \end{aligned}$$

Theorem (Hamza, Kaspi, K.)

Let $T_\varepsilon = \inf\{t : y(t) = \alpha\}$. *Then there exists* $\lim_{\varepsilon\to\infty} \mathbf{x}(T_\varepsilon + t) = \mathbf{x}_\alpha(t)$, *that uniquely solves*

$$\mathbf{x}_\alpha(t) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} + \int_0^t \mathbf{G}(\mathbf{x}_\alpha(u))du,$$

where $\beta = z(\alpha)$ *and* $z(u)$ *solves d.e.* $z'(u) = \Pi(z(u), u)$, $z(0) = a$, *with*

$$\Pi(z, u) = \frac{z(a - z - \gamma u)}{u(b - \gamma z - u)}$$

Approximations

The ‘correct’ approximation is given first by a Branching Markov process, and then by the solution of the deterministic equation.

Branching approximation
Let $\beta = b - \gamma a$.
The branching approximation holds *in total variation* up to a time $\tau^Y_{N,\alpha}$, chosen so that $NY(\tau^Y_{N,\alpha})$ is approximately $N^{1-\alpha}$.
It is not accurate for $\alpha \leq 1/3$; so we take $\alpha = 5/12$.
If the branching process is absorbed in 0, then so too, with high probability, is Y.
If not, then we show that $x_N(\tau^Y_N)$ is close to $x(t_N)$, where $t_N = \beta^{-1}\frac{7}{12}\ln N$, approximately the time when deterministic solution y reaches $N^{\frac{7}{12}}$.
By linearization near fixed point the time to reach $N^{\frac{7}{12}}$ is $t_N + O(1)$
Thus x_N closely follows the deterministic path, but with a random time shift.

Theorem

Theorem (Barbour, Hamza, Kaspi, K.(2015))

The process Z is a linear birth and death process, with per capita birth and death rates b and γa respectively.
$W = \lim_{t\to\infty} Z(t)e^{-\beta t}$. *Then except on an event E^c_{N1} of asymptotically negligible probability, the paths of NY and of Z can be coupled so as to be identical until the time* $\min\{\tau^Z(0), \tau^Z_{N,5/12}\}$, *in which case*
$\tau^Z_{N,5/12} = \tau^Y_{N,5/12} = \beta^{-1}\{\frac{7}{12}\ln N - \ln W\} + O(N^{-7/48})$.
For any T there exists a constant $\gamma > 0$, a constant $k_T < \infty$ and an event E^T_{N2} such that, on $\{\tau^Y_{N,5/12} < \infty\} \cap E_{N1} \cap E^T_{N2}$,

$$\sup_{0\le t\le \frac{5}{12}\beta^{-1}\ln N+T} |x_N(\tau^Y_{N,5/12} + t) - x(t_N + t)| \le k_T N^{-\gamma},$$

and $\lim_{N\to\infty} \mathbb{P}[E^T_{N2} \,|\, \{\tau^Z_{N1} < \infty\} \cap E_{N1}] = 1$.

THANK YOU!