On critical branching processes with immigration in varying environment

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Outline

- Sequence of branching processes with immigration (BPI)
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Sequence of branching processes with immigration

Galton–Watson branching processes with immigration (BPI)

$$\boxed{X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)}, \qquad k, n \in \mathbb{N}, \qquad X_0^{(n)} = 0,}$$

where, for each $n \in \mathbb{N}$, both the offsprings $\{\xi_{k,j}^{(n)} : k, j \in \mathbb{N}\}$ and the immigrations $\{\varepsilon_k^{(n)} : k \in \mathbb{N}\}$ are identically distributed, and they are independent, nonnegative, integer valued random variables.

Parameters: $m_n := \mathsf{E}\xi_{1,1}^{(n)}, \quad \lambda_n := \mathsf{E}\varepsilon_1^{(n)},$

$$\sigma_n^2 := \mathrm{Var}\xi_{1,1}^{(n)}, \quad b_n^2 := \mathrm{Var}\varepsilon_1^{(n)}.$$

Classification:	$m_n < 1$	$m_n = 1$	$m_n > 1$
	subcritical	**critical**	**supercritical**

Asymptotic result I: strictly positive offspring variance

Sriram AS (1994)

Suppose that $m_n = 1 + \alpha n^{-1} + o(n^{-1})$ with $\alpha \in \mathbb{R}$, and $\sigma_n^2 \to \sigma^2 > 0$ as $n \to \infty$. Let $\mathcal{X}_t^n := X_{\lfloor nt \rfloor}^{(n)}$. Then

$$\boxed{n^{-1}\mathcal{X}^n \xrightarrow{\mathcal{L}} \mathcal{X} \qquad \text{as} \qquad n \to \infty,}$$

where $\mathcal{X} := (\mathcal{X}_t)_{t \in \mathbb{R}_+}$ is a (nonnegative) diffusion process with initial value $\mathcal{X}_0 = 0$ and with generator

$$Lf(x) = (\lambda + \alpha x) f'(x) + \frac{1}{2}\sigma^2 x f''(x), \qquad f \in C_c^\infty(\mathbb{R}_+).$$

This process can also be characterized as the unique solution to the SDE

$$\boxed{\mathrm{d}\mathcal{X}_t = (\lambda + \alpha \mathcal{X}_t)\,\mathrm{d}t + \sigma\sqrt{(\mathcal{X}_t)_+}\,\mathrm{d}\mathcal{W}_t, \quad t \in \mathbb{R}_+, \quad \mathcal{X}_0 = 0.}$$

This is a square-root process or a CBI process, related to the squared Bessel process and Cox-Ingersoll-Ross model in the financial mathematics.

Asymptotic result II: vanishing offspring variance

I, Pap, van Zuijlen JAP (2005)

Suppose that $m_n = 1 + \alpha n^{-1} + o(n^{-1})$ with $\alpha \in \mathbb{R}$, and $\sigma_n^2 = \beta n^{-1} + o(n^{-1})$ as $n \to \infty$. Let $\mathcal{M}_t^n := \sum_{k=1}^{\lfloor nt \rfloor} M_k^n$.
Then, we have fluctuation limit theorem

$$\boxed{n^{-1/2}(\mathcal{X}^n - \mathrm{E}\mathcal{X}^n, \mathcal{M}^n) \xrightarrow{\mathcal{L}} (\mathcal{X}, \mathcal{M}) \qquad \text{as} \qquad n \to \infty,}$$

where $\mathcal{M} := (\mathcal{M}_t)_{t \in \mathbb{R}_+}$ is a time–changed Wiener process, i.e., $\mathcal{M}_t = W_{T(t)}$, $t \in \mathbb{R}_+$, with

$$T(t) := b^2 t + \beta\lambda \int_0^t \int_0^s \mathrm{e}^{\alpha u} \mathrm{d}u \mathrm{d}s,$$

$(W_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process; and

$$\boxed{\mathcal{X}_t := \int_0^t \mathrm{e}^{\alpha(t-s)} \mathrm{d}\mathcal{M}_s}$$

is an Ornstein–Uhlenbeck process driven by $\mathcal{M}$.

BPI in varying environment (BPIVE)

Goal: To study the nearly criticality in one model!
Galton–Watson branching process with immigration

$$\boxed{X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \mathbb{N}, \qquad X_0 = 0,}$$

where the offsprings $\{\xi_{k,j} : k, j \in \mathbb{N}\}$ are identically distributed for each $k \in \mathbb{N}$, respectively, and they and the immigrations $\{\varepsilon_k : k \in \mathbb{N}\}$ are independent, nonnegative, integer valued random variables. The offspring and the immigration distributions may vary from generation to generation.

The process $(X_k)_{k \in \mathbb{Z}_+}$ is called branching process with immigration in varying environment (BPIVE) or time varying BPI.

Parameters: $m_k := \mathsf{E}\xi_{k,1}$, $\lambda_k := \mathsf{E}\varepsilon_k$, $\sigma_k^2 := \mathrm{Var}\xi_{k,1}$, $b_k^2 := \mathrm{Var}\varepsilon_k$.

Applications

General inhomogeneous branching processes:

- Domain of peer-to-peer file sharing networks, Adar and Huberman (2000), Zhao et al. (2005)
- Modeling biodiversity or macroevolution, Aldous and Popovic (2005), Haccou and Iwasa (1996)
- Epidemic–type Aftershock Sequence (ETAS) in seismology, Farrington et al. (2003)

Heterogeneous INAR models (Bernoulli offsprings):

- Understanding and predicting consumers' buying behaviour, Böckenholt (1999)
- Modeling the premium in bonus-malus scheme of car insurance, Gourieroux and Jasiak (2004)

Asymptotic for the mean

We have the following deterministic time varying linear recursion for the mean

$$\boxed{\mathsf{E}(X_k) = m_k \mathsf{E}(X_{k-1}) + \lambda_k, \qquad k \in \mathbb{N},}$$

where the sequence $(m_k)_{k\in\mathbb{N}}$ determines the asymptotic behaviour of the process.

Define the bottom and top Lyapunov exponents as

$$\sup_n n^{-1} \inf_k r(n,k) =: \gamma_b \quad \leq \quad \gamma_t := \inf_n n^{-1} \sup_k r(n,k),$$

where the partial growing rate function is defined as

$$r(n,k) := \sum_{j=k}^{k+n-1} \log m_j$$

Classification:	$\gamma_t < 0$	$\gamma_b \leq 0 \leq \gamma_t$	$\gamma_b > 0$
	subcritical	**???**	**supercritical**

The supercritical case was studied by Goettge (1976), Cohn and Hering (1983), Jagers and Nerman (1985), D'Souza and Biggins (1992, 1993), D'Souza (1994).

Asymptotic for nearly critical recursion

Notation: $\lambda_k \rightsquigarrow \lambda$ stands for the Cesaro convergence $n^{-1}\sum_{k=1}^{n} \lambda_k \to \lambda$ as $n \to \infty$.

For a deterministic time varying linear recursion

$$\boxed{x_k = m_k x_{k-1} + \lambda_k, \qquad k \in \mathbb{N}, \qquad x_0 = 0,}$$

where

1. $m_k = 1 + \alpha k^{-1} + \delta_k$ for some $\alpha \in \mathbb{R}$ and $\sum_{k=1}^{\infty} |\delta_k| < \infty$;
2. $\lambda_k \rightsquigarrow \lambda$ as $k \to \infty$ for some $\lambda \geq 0$, where $(\lambda_k)_{k \in \mathbb{Z}_+}$ is a non-negative sequence,

we have

1. $n^{-1} x_n \to \lambda(1-\alpha)^{-1}$ if $\alpha < 1$, i.e. $EX_n = O(n)$;
2. $(n \ln n)^{-1} x_n \to \lambda$ if $\alpha = 1$, i.e. $EX_n = O(n \ln n)$;
3. $n^{-\alpha} x_n \to \kappa$ if $\alpha > 1$, i.e. $EX_n = O(n^{\alpha})$,
   where $\kappa > 0$ is a constant,

as $n \to \infty$.

Proof of the asymptotic

Representation for the unique solution, see Elaydi (1.2.4),

$$x_k = \sum_{j=1}^{k} \prod_{i=j+1}^{k} m_i \lambda_j$$

Method of the proof: perturbation argument.

Introduce the new sequence $y_k := k^{-\alpha} x_k$, $k \in \mathbb{N}$. Then, we have

$$\boxed{y_k = (1 + \widetilde{\delta}_k) y_{k-1} + k^{-\alpha} \lambda_k,}$$

where $\sum_{k=1}^{\infty} |\widetilde{\delta}_k| < \infty$.

This recursion is a small perturbation of the recursion

$$\boxed{z_k = z_{k-1} + k^{-\alpha} \lambda_k.}$$

Hence, their asymptotic behaviors are similar. Thus, we have

$$\boxed{x_n \approx n^{\alpha} \sum_{k=1}^{n} k^{-\alpha} \lambda_k}$$

Proof of the asymptotic

In case of $\alpha < 1$, by Toeplitz theorem, we have

$$n^{-1}x_n \approx n^{\alpha-1}\sum_{k=1}^{n} k^{-\alpha}\lambda_k \to \frac{\lambda}{1-\alpha}$$

since

$$n^{\alpha-1}\sum_{k=1}^{n} k^{-\alpha} \approx \int_0^1 s^{-\alpha} ds = (1-\alpha)^{-1}$$

Second order linear difference equations I.

An inhomogeneous first order d.e. can be transformed to a homogeneous second order d.e. Suppose that

$$m_k = 1 + \alpha k^{-1} + \beta k^{-2}, \qquad \lambda_k = \lambda + \gamma k^{-1}.$$

Then

$$\boxed{x_k + a(k)x_{k-1} + b(k)x_{k-2} = 0, \qquad k = 2, 3, \ldots,}$$

where

$$a(k) \approx a_0 + a_1 k^{-1} + a_2 k^{-2}, \qquad b(k) \approx b_0 + b_1 k^{-1} + b_2 k^{-2}$$

with $a_0 = -2$, $a_1 = -\alpha$, $a_2 = \gamma - \beta$
and $b_0 = 1$, $b_1 = \alpha$, $b_2 = \alpha + \beta - \gamma$.
Asymptotic theory: Wong and Li (1992), goes back to Birkhoff ('30).

Second order linear difference equations II.

Characteristic polynomial:

$$\varrho^2 + a_0\varrho + b_0 \implies \varrho^2 - 2\varrho + 1 = 0 \implies \varrho_{1,2} = 1$$

The common value is a root of the auxiliary equation

$$a_1\varrho + b_1 = 0 \implies -\alpha\varrho + \alpha = 0$$

Indicial polynomial:

$$\kappa(\kappa - 1)\varrho^2 + (a_1\kappa + a_2) + b_2 \implies \kappa^2 - (\alpha + 1)\kappa + \alpha = 0$$

Roots: $\kappa_1 = 1$ and $\kappa_2 = \alpha$.
Two linearly independent asymptotic solutions if $\alpha \neq 1$ with

$$x_n^{(i)} \approx \varrho^n n^{\kappa_i} \qquad i = 1, 2$$

If $\alpha = 1$ then we have an extra log n factor.

Criticality and classification

A BPIVE is called asymptotically critical if $m_n \to 1$ as $n \to \infty$. More precisely, if the parametrization

$$m_k = 1 + \alpha k^{-1} + \delta_k, \qquad \alpha \in \mathbb{R}, \qquad \sum_{k=1}^{\infty} |\delta_k| < \infty$$

holds then BPIVE is called nearly critical and it has criticality index α. In this case, $\gamma_b = \gamma_t = 0$.

Classification (regimes) for nearly critical TVBPI:

$\alpha < 1$	$\alpha = 1$	$\alpha > 1$	
proper	**logarithmically**	**polinomially**	**nearly critical**

In the sequel, we investigate proper nearly critical BPIVE.
If $\alpha = 0$ then a BPIVE is called strongly critical.

Limit theorem: Assumptions I (2015)

Suppose that

(i) $m_n = 1 + \alpha n^{-1} + \delta_n$ with $\alpha < 1$ and $\sum_{n=1}^{\infty} |\delta_n| < \infty$;

(ii) $\lambda_n \rightsquigarrow \lambda \geq 0$ as $n \to \infty$;

(iii) $\sigma_n^2 \rightsquigarrow \sigma^2 \geq 0$ as $n \to \infty$;

(iv) $n^{-1} b_n^2 \rightsquigarrow 0$ as $n \to \infty$;

moreover the following Lindeberg conditions hold

$$\text{(L1)} \quad \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}\left(|\xi_{k,1} - m_k|^2 \mathbb{1}_{\{|\xi_{k,1} - m_k| > \theta n\}} \right) \to 0 \text{ for all } \theta > 0,$$

$$\text{(L2)} \quad \frac{1}{n^2} \sum_{k=1}^{n} \mathsf{E}\left(|\varepsilon_k - \lambda_k|^2 \mathbb{1}_{\{|\varepsilon_k - \lambda_k| > \theta n\}} \right) \to 0 \text{ for all } \theta > 0$$

as $n \to \infty$.

Limit theorem: Result I (2015)

Let $\mathcal{X}_t^n := X_{\lfloor nt \rfloor}$. Then, weakly in the Skorokhod space $\mathrm{D}(\mathbb{R}_+, \mathbb{R})$,

$$\boxed{n^{-1}\mathcal{X}^n \xrightarrow{\mathcal{L}} \mathcal{X} \qquad \text{as} \qquad n \to \infty,}$$

where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ satisfies the SDE

$$\boxed{\mathrm{d}\mathcal{X}_t = (\lambda + \alpha t^{-1}\mathcal{X}_t)\,\mathrm{d}t + \sigma\sqrt{\mathcal{X}_t}\mathrm{d}\mathcal{W}_t, \qquad t > 0,}$$

where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process, with initial condition $\mathcal{X}_0 = 0$.

Formal SDE: the drift $\beta(t, x) := (\lambda + \alpha t^{-1} x)$ is not Lipschitz and it is not defined at $t = 0$.

Solution: take the process $\mathcal{Y}_t := t^{-\alpha}\mathcal{X}_t$ and apply the Ito's formula (formally)

$$\boxed{\mathrm{d}\mathcal{Y}_t = \lambda t^{-\alpha}\,\mathrm{d}t + \sigma\sqrt{t^{-\alpha}\mathcal{Y}_t}\mathrm{d}\mathcal{W}_t, \qquad t > 0,}$$

Associated martingale differences

Let $\mathcal{F}_k$ denote the σ-algebra generated by $X_0, X_1, \ldots, X_k$. We have the conditional expectation

$$\mathrm{E}(X_k \mid \mathcal{F}_{k-1}) = m_k X_{k-1} + \lambda_k. \qquad k \in \mathbb{N}.$$

Clearly,

$$M_k := X_k - \mathrm{E}(X_k \mid \mathcal{F}_{k-1}) = X_k - m_k X_{k-1} - \lambda_k$$

defines a martingale difference sequence with respect to the filtration $(\mathcal{F}_k)_{k \in \mathbb{Z}_+}$. On the other hand,

$$M_k := \sum_{j=1}^{X_{k-1}} (\xi_{k,j} - m_k) + \varepsilon_k - \lambda_k$$

Thus, we have the heteroscedastic property:

$$\mathrm{E}(M_k^2 \mid \mathcal{F}_{k-1}) = \sigma_k^2 X_{k-1} + b_k^2$$

Heuristic for the limit theorem I.

For all $k \in \mathbb{N}$, we have

$$X_k = m_k X_{k-1} + \lambda_k + M_k = X_{k-1} + \lambda_k + \alpha \frac{X_{k-1}}{k} + \delta_k X_{k-1} + M_k$$

Let $0 < s < t$. Then, by iteration,

$$\frac{X_{\lfloor nt \rfloor}}{n} = \frac{X_{\lfloor ns \rfloor}}{n} + \frac{1}{n} \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \left(\lambda_k + \alpha \frac{X_{k-1}}{k} \right) + \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \sigma_k \sqrt{\frac{X_{k-1}}{n}} W_{n,k}$$

$$\frac{1}{n} \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} (\delta_k X_{k-1} + \varepsilon_k - \lambda_k)$$

where

$$W_{n,k} := \frac{1}{\sqrt{n X_{k-1}}} \sum_{j=1}^{X_{k-1}} \frac{\xi_{k,j} - m_k}{\sigma_k} \approx \mathcal{N}(1, n^{-1})$$

Heuristic for the limit theorem II.

The blue part can be approximated by stochastic integral equation

$$\boxed{\mathcal{X}_t = \mathcal{X}_s + \int_s^t \left(\lambda + \alpha\frac{\mathcal{X}_u}{u}\right) du + \int_s^t \sigma\sqrt{\mathcal{X}_u} d\mathcal{W}_u}$$

On the other hand, the red part is vanishing in probability since

$$\frac{1}{n}\sum_{k=1}^{n} |\delta_k| \mathrm{E}(X_{k-1}) \le \frac{C}{n}\sum_{k=1}^{n} |\delta_k| k \to 0$$

by Kronecker lemma and, by assumption (iv),

$$\mathrm{Var}\left(\frac{1}{n}\sum_{k=1}^{n}(\varepsilon_k - \lambda)\right) = \frac{1}{n^2}\sum_{k=1}^{n} b_k^2 \to 0$$

Weak convergence to a diffusion process I.

For each $n \in \mathbb{N}$, let $(U_k^n)_{k\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$–valued adapted random variables w.r.t. a filtration $(\mathcal{F}_k^n)_{k\in\mathbb{Z}_+}$. Introduce the random step functions:

$$\boxed{\mathcal{U}_t^n := \sum_{k=1}^{\lfloor nt\rfloor} U_k^n\,, \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.}$$

Let $(\mathcal{U}_t)_{t\in\mathbb{R}_+}$ be a d–dimensional diffusion process

$$\boxed{\mathrm{d}\mathcal{U}_t = \beta(t,\mathcal{U}_t)\,\mathrm{d}t + \gamma(t,\mathcal{U}_t)\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,}$$

where $\beta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ are continuous functions and $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is an r-dimensional standard Wiener process.
Assume that the SDE has a unique weak solution with $\mathcal{U}_0 = x_0$ for all $x_0 \in \mathbb{R}^d$. Let $(\mathcal{U}_t)_{t\in\mathbb{R}_+}$ be a solution with $\mathcal{U}_0 = 0$.

Weak convergence to a diffusion process II.

I & Pap, (2010)

Suppose that, for each $T > 0$,

Uniform convergence on compacts in probability (ucp)

$$\sup_{t\in[0,T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \mathrm{E}\big(U_k^n \,\big|\, \mathcal{F}_{k-1}^n\big) - \int_0^t \beta(s,\mathcal{U}_s^n)\,\mathrm{d}s \right\| \xrightarrow{\mathrm{P}} 0,$$

$$\sup_{t\in[0,T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \mathrm{E}\big(U_k^n (U_k^n)^\top \,\big|\, \mathcal{F}_{k-1}^n\big) - \int_0^t \gamma(s,\mathcal{U}_s^n)\gamma(s,\mathcal{U}_s^n)^\top\,\mathrm{d}s \right\| \xrightarrow{\mathrm{P}} 0,$$

and the conditional Lindeberg condition

$$\sum_{k=1}^{\lfloor nT \rfloor} \mathrm{E}\big(\|U_k^n\|^2 \mathbb{1}_{\{\|U_k^n\|>\theta\}} \,\big|\, \mathcal{F}_{k-1}^n\big) \xrightarrow{\mathrm{P}} 0 \quad \text{for all } \theta > 0.$$

Then

$$\boxed{\mathcal{U}^n \xrightarrow{\mathcal{L}} \mathcal{U} \qquad \text{as} \qquad n \to \infty.}$$

Sketch for the proof of the limit theorem

Introduce the new process $Y_k := k^{-\alpha} X_k$, $k \in \mathbb{N}$, $Y_0 := 0$ and define

$$\boxed{U_k^n := n^{\alpha-1}(Y_k - Y_{k-1}), \qquad k, n \in \mathbb{N}.}$$

Then

$$\mathcal{U}_t^n := \sum_{k=1}^{\lfloor nt \rfloor} U_k^n = n^{\alpha-1} Y_{\lfloor nt \rfloor}$$

We prove, by general limit theorem, that weakly in the Skorokhod space $\mathsf{D}(\mathbb{R}_+, \mathbb{R})$

$$n^{\alpha-1} Y_{\lfloor nt \rfloor} = \mathcal{U}_t^n \xrightarrow{\mathcal{L}} \mathcal{U}_t := \mathcal{Y}_t \qquad \text{as} \qquad n \to \infty$$

Sketch for the proof of the limit theorem

This implies, for $0 \leq t_1 < t_2 < \ldots < t_m$,

$$n^{\alpha-1}\left(Y_{\lfloor nt_1 \rfloor}, \ldots, Y_{\lfloor nt_m \rfloor}\right) \xrightarrow{\mathcal{L}} (\mathcal{Y}_{t_1}, \ldots, \mathcal{Y}_{t_m})$$

Hence

$$n^{-1}\left(X_{\lfloor nt_1 \rfloor}, \ldots, X_{\lfloor nt_m \rfloor}\right) \xrightarrow{\mathcal{L}} (\mathcal{X}_{t_1}, \ldots, \mathcal{X}_{t_m})$$

shows the convergence of finite dimensional distributions. Then, we prove tigthness by checking the conditional Lindeberg condition.

Deterministic limit theorem I (2015)

Suppose that

1. $m_n = 1 + \alpha n^{-1} + \delta_n$ with $\alpha < 1$ and $\sum_{n=1}^{\infty} |\delta_n| < \infty$;
2. $\lambda_n \rightsquigarrow \lambda \geq 0$ as $n \to \infty$;
3. $\sigma_n^2 \rightsquigarrow 0$, as $n \to \infty$;
4. $n^{-1} b_n^2 \rightsquigarrow 0$ as $n \to \infty$.

Then

$$\boxed{n^{-1} \mathcal{X}^n \xrightarrow{\mathcal{L}} \mu_{\mathcal{X}} \qquad \text{as} \qquad n \to \infty,}$$

where $\mu_{\mathcal{X}} : \mathbb{R}_+ \to \mathbb{R}_+$ is the unique solution of the ordinary differential equation (ODE)

$$\boxed{\mathrm{d}\mu_{\mathcal{X}}(t) = (\lambda + \alpha t^{-1} \mu_{\mathcal{X}}(t))\mathrm{d}t, \qquad t > 0,}$$

with initial condition $\mu_{\mathcal{X}}(0) = 0$. In fact, $\mu_{\mathcal{X}}(t) = \lambda t/(1 - \alpha)$.

Fluctuation limit theorem: Assumptions I (2015)

Suppose that

(i) $m_n = 1 + \alpha n^{-1} + \delta_n$ with $\alpha < 1$ and $\sum_{n=1}^{\infty} |\delta_n| < \infty$;

(ii) $\lambda_n \rightsquigarrow \lambda \geq 0$ as $n \to \infty$;

(iii) $n\sigma_n^2 \rightsquigarrow \sigma^2 \geq 0$, as $n \to \infty$;

(iv) $b_n^2 \rightsquigarrow b^2 \geq 0$ as $n \to \infty$;

moreover the following Lindeberg conditions hold

(L1) $\sum_{k=1}^{n} \mathrm{E}\left(|\xi_{k,1} - m_k|^2 \mathbb{1}_{\{|\xi_{k,1} - m_k| > \theta n^{1/2}\}}\right) \to 0$ for all $\theta > 0$,

(L2) $\frac{1}{n}\sum_{k=1}^{n} \mathrm{E}\left(|\varepsilon_k - \lambda_k|^2 \mathbb{1}_{\{|\varepsilon_k - \lambda_k| > \theta n^{1/2}\}}\right) \to 0$ for all $\theta > 0$

as $n \to \infty$.

Fluctuation limit theorem: Statements

Then, weakly in the Skorokhod space $\mathsf{D}(\mathbb{R}_+, \mathbb{R})$,

$$\boxed{n^{-1/2}\mathcal{M}^n \xrightarrow{\mathcal{L}} \mathcal{M} \qquad \text{as} \qquad n \to \infty,}$$

where $(\mathcal{M}_t)_{t\in\mathbb{R}_+}$ is a Wiener process with variance

$$\sigma^2_{\mathcal{M}} := \sigma^2 \frac{\lambda}{1-\alpha} + b^2.$$

Moreover, suppose that $\alpha < 1/2$. Then, weakly in the Skorokhod space $\mathsf{D}(\mathbb{R}_+, \mathbb{R}^2)$,

$$\boxed{n^{-1/2}(\mathcal{X}^n - \mathsf{E}\mathcal{X}^n, \mathcal{M}^n) \xrightarrow{\mathcal{L}} (\mathcal{X}, \mathcal{M}) \qquad \text{as} \qquad n \to \infty,}$$

where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ satisfies the SDE

$$\boxed{\mathrm{d}\mathcal{X}_t = \alpha t^{-1}\mathcal{X}_t \mathrm{d}t + \mathrm{d}\mathcal{M}_t, \qquad t > 0,}$$

with initial condition $\mathcal{X}_0 = 0$.

Ornstein-Uhlenbeck fluctuation limit

The SDE can be written in the form

$$\mathrm{d}\mathcal{X}_t = \alpha t^{-1}\mathcal{X}_t \mathrm{d}t + \sigma_{\mathcal{M}} \mathrm{d}W_t, \qquad t > 0.$$

The solution is given as

$$\boxed{\mathcal{X}_t = \sigma_{\mathcal{M}} t^{\alpha} \int_0^t s^{-\alpha} \mathrm{d}W_s, \qquad t > 0.}$$

If $\alpha < 1/2$ the integral is well-defined in L^2 and Itô's sense as well since

$$\int_0^t s^{-2\alpha} \mathrm{d}s = \frac{t^{1-2\alpha}}{1-2\alpha} < \infty.$$

$(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is an Ornstein–Uhlenbeck type process

$$\boxed{\mathcal{X}_t = \sigma_{\mathcal{M}} \int_0^t e^{\alpha(\ln t - \ln s)} \mathrm{d}W_s, \qquad t > 0,}$$

with logarithmic exponent function.

Why $\alpha < 1/2$?

Define the sequence $V_k := \mathrm{Var}(X_k)$, $k \in \mathbb{N}$. Then we have the recursion

$$\boxed{V_k = m_k^2 V_{k-1} + \mathsf{E} M_k^2, \qquad k \in \mathbb{N}.}$$

Asymptotic for the variance:

1. $n^{-1} V_n \to \lambda(1-2\alpha)^{-1}((1-\alpha)^{-1}\lambda\sigma^2 + b^2)$ if $\alpha < 1/2$,
2. $(n \ln n)^{-1} V_n \to (1-\alpha)^{-1}\lambda\sigma^2 + b^2$ if $\alpha = 1/2$,
3. $n^{-2\alpha} V_n \to c \geq 0$ if $\alpha > 1/2$, where $c \in \mathbb{R}$ is a constant,

since

$$m_k^2 = 1 + 2\alpha k^{-1} + \widetilde{\delta}_k, \qquad k \in \mathbb{N},$$

with $\sum_{k=1}^{\infty} |\widetilde{\delta}_k| < \infty$.

Conclusions and future works

- A criticality index was introduced for branching processes with immigration in varying environment.
- Limit theorems were proved for proper nearly critical branching processes with immigration in varying environment.
- Estimating the criticality index and the average immigration intensity.
- Developing tests for hypothesis $H_0 : \alpha = \alpha_0$, e.g., testing the strong criticality $(\alpha_0 = 0)$.
- Investigating the logarithmically and polinomially nearly critical BPIVE.

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Thank you for your attention!