The standard Galton-Watson branching process with a reflecting barrier

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Description of the reflection process.

This barrier will be identified by 2 predetermined positive integers j and k where $j \leq k$. If in any generation, the population size exceeds k, only j of those individuals will have the capacity to reproduce. So, if there are more than k individuals in a generation, the process immediately starts over with a population j individuals. Call k, the barrier value and j the restart value.

Dr. Raymond Pearl quotation, 1939

It is likely that biology will eventually be as fully expressed in mathematical theory as physics now is. The process is already started and …

More on Pearl quotation

There is no substitute for mathe-
matics to state in rational shorthand
the relations between natural
phenomena or generalizations about
them.

Who started the revival?

In 2003,a report (Bio 2010) from the National Academy of Sciences (NAS) set a goal of having biology undergraduates be quantatively literate by 2010. (A list of 8 quant. topics followed.)

Math and Bio 2010, Math. Assoc. - America

M and B 2010 sees the disciplines of mathematics and biology, currently quite separate, will be linked in the undergraduate science programs of the twenty-first century.

M and B chapter title

The “Gift” of Mathematics in the Era of Biology

MAA Focus Newsmagazine Feb./March 2015

When biology was primarily a descriptive science, biologists had little need of mathematical services. But a revolution has occurred-more precisely, is occurring-in biology, the discipline is becoming mathematical

A template for objections

Branching processes are not relevant models for biology because

______________________________.

(Fill in the blank with the objection.)

The template completed

Branching processes are not relevant models for biology because survival requires that there are no upper limits on the population sizes of the ongoing generations.

Ecosystem Analysis and Prediction

Proceedings of a conf. on Ecosystems

Alta, Utah USA July 1 – 5, 1974

COMMENTS FROM A BIOLOGIST TO A MATHEMATICAN (pp. 318 – 329)

Lawrence B. Slobodkin (1928-2009)

The latter half of the opening paragraph.

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I have discovered that there are ten things I very much wish that math. would stop doing in pop. biology

Item #3 in the list of ten negative actions

I wish they would not build models in which feedback terms are completely absent. It is obvious from Random Walk Theory, that a biological system moving through time

More on Item #3

will either explode, become extinct, or reach some elaborate discontin-uity unless it has feedback controls built into it somewhere along the line. This is an old argument and I

Still more on item #3

think that we have the situation sufficiently well settled so that non-feedback models are simply not legitimate (7).

Some questions re: the previous statement.

1. What does he know about ran-dom walks? about branching processes?
2. What is the reference that rules out all non-feedback models?

Question 3

3. What is the meaning of "feedback"?

Feedback from Wikipedia

Feedback occurs when outputs of a system are “fed back” as inputs in a chain of cause and effect state-ments that form a circuit or loop.

Standard process example

Binary Splitting where

$$X_0 = 1$$

$$p_0 = p_2 = 1/2$$

An acceptable feedback term

The phrase "to feed back" can be used in the sense of "returning to an earlier position"

Example: $k = 5$ and $j = 4$.

Some obvious characteristics of the example

In any generation, starting with n =1, the population size will be 0, 2, or 4. This is a finite state Markov chain with a single absorbing state. The extinction probability equals one.

A system of equations describing the process

$$P(X_n = 0) = P(X_{n-1}=0)$$
$$+ P(X_n=0|X_{n-1}=2)P(X_{n-1}=2) + P(X_n=0|X_{n-1}=4)P(X_{n-1}=4)$$

$$P(X_n=2) = P(X_n=2|X_{n-1}=2)P(X_{n-1}=2) + P(X_n=2|X_{n-1}=4)P(X_{n-1}=4)$$

$$P(X_n=4) = P(X_n=4|X_{n-1}=2)P(X_{n-1}=2) + P(X_n=4|X_{n-1}=4)P(X_{n-1}=4)$$

A system of equations defining the process

$$P(X_n=0) = P(X_{n-1}=0)$$
$$+ (1/4)P(X_{n-1}=2) + (1/16)P(X_{n-1}=4)$$
$$P(X_n=2) = (2/4)P(X_{n-1}=2) + (4/16)P(X_{n-1}=4)$$
$$P(X_n=4) = (1/4)P(X_{n-1}=2) + (11/16)P(X_{n-1}=4)$$

Matrix form of the equations defining process

$$\begin{bmatrix} P(Xn = 0) \\ P(Xn = 2) \\ P(Xn = 4) \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & 1/16 \\ 0 & 2/4 & 4/16 \\ 0 & 1/4 & 11/16 \end{bmatrix} \begin{bmatrix} P(Xn-1 = 0) \\ P(Xn-1 = 2) \\ P(Xn-1 = 4) \end{bmatrix}$$

Matrix form of the equations defining process

$$\text{Let } M = \begin{bmatrix} 1 & 1/4 & 1/16 \\ 0 & 2/4 & 4/16 \\ 0 & 1/4 & 11/16 \end{bmatrix}$$

Some more notation

$$\text{Let } a(n) = \begin{bmatrix} P(Xn = 0) \\ P(Xn = 2) \\ P(Xn = 4) \end{bmatrix}$$

Calculations involving powers of M

Then $a(2) = Ma(1)$,

$a(3) = M^2a(1)$,

……………………

$a(n) = M^{n-1}a(1)$.

A shorter approach for $P(X_n = i)$, $i = 2, 4$

Let R (survival matrix) = $\begin{matrix} 2/4 & 4/16 \\ 1/4 & 11/16 \end{matrix}$

R has eigenvalue .8608 with eigen-vector v=(.4093,.5907) and eigenval. .3268 with eigenvector u=(1, -.6930)

$P(X_2 = 2) = 1/4$ and $P(X_2 = 4) = 1/8$

This comes from (1/2) multiplied by the first column of R.

$$\begin{bmatrix} 2/4 & 4/16 \\ 1/4 & 11/16 \end{bmatrix} * \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/8 \end{bmatrix}$$

Consider the 3rd generation.

The survival probabilities come from ½ multiplied by the first column of R^2 which is ½ times R multiplied by C1 of R, i.e. (1/2)xRx[.6822v + .2207u]

= (1/2)x[.6822Rv + .2207Ru]

More on the 3rd generation.

Since v and u are eigenvectors, the survival probabilities are

$(1/2)x[.6822(.8608v)+.2207(.3268u)]$

nth generation survival probabilities

Again, since u and v are eigenvalues, the nth generation survival probabilities are (1/2) of $[.6822(.8608^{n-2}v)+.2207(.3268^{n-2}u)]$.

Use of Agresti paper to estimate T

Reference: Bounds on the extinction time distribution of a branching process. Adv. Appl. Prob. **6**, 332-335 (1974)