Branching Structures within Random walks on Z and their Applications

Joint work with Prof. Wen-Ming Hong

Hua-Ming Wang
(Anhui Normal University, P. R. China)

III Workshop on Branching Processes and their Applications, University of Extremadura

7 April 2015

目 录

Branching process in (1,1) RW

- Harris (1952),
  (1,1) random walk $\leftrightarrow$ Galton-Watson tree.

Branching process in (1,1) RW

- Harris (1952),
  (1,1) random walk $\leftrightarrow$ Galton-Watson tree.
- Dwass (1975),
  Branching process (with immigration) in simple RW.

Branching process in (1,1) RW

- Harris (1952),
  (1,1) random walk $\leftrightarrow$ Galton-Watson tree.
- Dwass (1975),
  Branching process (with immigration) in simple RW.
- Kesten, Kozlov and Spitzer (1975),
  Branching process $\leftrightarrow$ non-homogeneous (1,1) RW;
  Stable law for RWRE.

Branching process in (1,1) RW

- Harris (1952),
  (1,1) random walk $\leftrightarrow$ Galton-Watson tree.
- Dwass (1975),
  Branching process (with immigration) in simple RW.
- Kesten, Kozlov and Spitzer (1975),
  Branching process $\leftrightarrow$ non-homogeneous (1,1) RW;
  Stable law for RWRE.
- Rogers (1984), by using branching structure, showed that local time of (1,1) RW converges to Brownian local time.

Branching process in (1,1) RW

- Harris (1952),
  (1,1) random walk $\leftrightarrow$ Galton-Watson tree.
- Dwass (1975),
  Branching process (with immigration) in simple RW.
- Kesten, Kozlov and Spitzer (1975),
  Branching process $\leftrightarrow$ non-homogeneous (1,1) RW;
  Stable law for RWRE.
- Rogers (1984), by using branching structure, showed that local time of (1,1) RW converges to Brownian local time.
- Gantert and Shi (2002), Many visit to a single site by RWRE.

Branching process in (1,1) RW

- Harris (1952),
  (1,1) random walk $\leftrightarrow$ Galton-Watson tree.
- Dwass (1975),
  Branching process (with immigration) in simple RW.
- Kesten, Kozlov and Spitzer (1975),
  Branching process $\leftrightarrow$ non-homogeneous (1,1) RW;
  Stable law for RWRE.
- Rogers (1984), by using branching structure, showed that local time of (1,1) RW converges to Brownian local time.
- Gantert and Shi (2002), Many visit to a single site by RWRE.
- Zeitouni (2004), Invariant density
  related to *"the environment viewed from particle"*.

Branching process in (1,1) RW

- Harris (1952),
  (1,1) random walk $\leftrightarrow$ Galton-Watson tree.
- Dwass (1975),
  Branching process (with immigration) in simple RW.
- Kesten, Kozlov and Spitzer (1975),
  Branching process $\leftrightarrow$ non-homogeneous (1,1) RW;
  Stable law for RWRE.
- Rogers (1984), by using branching structure, showed that local time of (1,1) RW converges to Brownian local time.
- Gantert and Shi (2002), Many visit to a single site by RWRE.
- Zeitouni (2004), Invariant density
  related to *"the environment viewed from particle"*.
- Afanasyev (2014) proved a conditional Ritter Theorem.

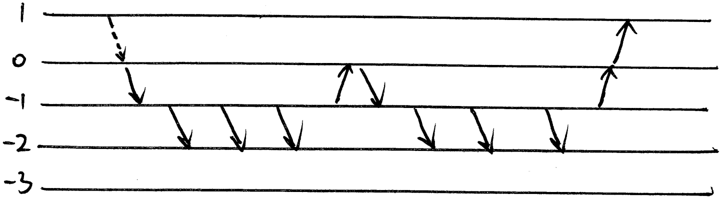
Branching Structure for (1,1) RW

Suppose $\limsup_{n\to\infty} X_n = \infty$. Define $T_1 = \inf\{n : X_n > 0\}$.
Let $U_0 = 1$, and for $i < 0$,

$$U_i = \#\{0 \le n < T_1 : X_n = i+1, X_{n+1} = i\}.$$

Then $U_0, U_{-1}, \ldots$ forms a branching process with

$$P(U_{i-1} = k | U_i = 1) = q_i^k p_i, \; k = 0, 1, 2, \ldots.$$

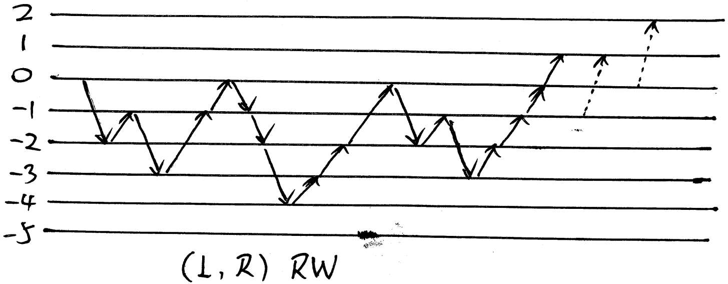


(1,1) RW

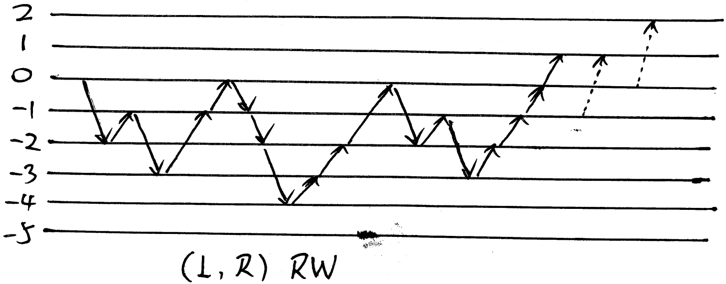
[KKS75] Kesten, H., Kozlov, M.V. and Spitzer, F., A limit law for random walk in a random environment, Compos. Math., Vol.30, pp 145-168, 1975

Branching structure for (L,R) RW? (Multitype branching process)

Branching structure for (L,R) RW? (Multitype branching process)



Branching structure for (L,R) RW?
(Multitype branching process)



Key indicated there maybe some multitype branching process within (L,1) RW path, but he did not give the construction.

[K84] Key, E.S., Limiting distributions and regeneration times for multitype branching processes within immigration in a random environment, Ann. Probab. 15(1), 344 - 353, pp 1987

- Hong and Wang (2013):
  Branching structure for (L,1) RW
- Hong and Zhang (2010):
  Branching structure for (1,R) RW
- Wang and Hong (2014):
  Branching structure for (L,R) RW

[HW13] Hong, W.M. and Wang, H.M., Intrinsic branching structure within (L-1) random walk in random environment and its applications, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, Vol. 16, 1350006 [14 pages], 2013

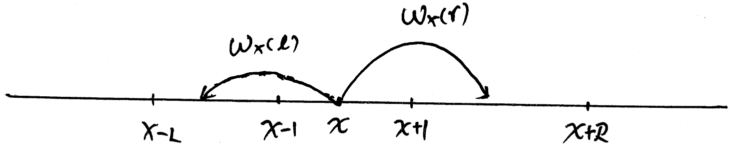
[HZ10] Hong, W.M. and Zhang, L., Branching structure for the transient (1; R)-random walk in random environment and its applications, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, Vol. 13(4), pp 589-618, 2010

[HW14] Wang, H.M. and Hong, W.M., Intrinsic branching structure within random walk on Z, *Theory Probab. Appl.*, Vol. 58(4), pp 640-659, 2014

(L,R) random walk

Fix $1 \leq L, R \in \mathbb{Z}$.
Let $\Lambda = \{-L, -L+1, ..., R\}/\{0\}$.
Environment: $\omega = (\omega_i)_{i\in\mathbb{Z}}$ where for $i \in \mathbb{Z}$, $\omega_i = (\omega_i(l))_{l\in\Lambda}$ is a probability measure on $i + \Lambda$.
Random walk $\{X_n\}$ **:** a Markov Chain, starting from 0, with transition probabilities

$$P_\omega(X_{n+1} = i + l | X_n = i) = \omega_i(l), \ l \in \Lambda.$$



Branching structure for (L,1) RW

Consider (L, 1) RW. Suppose $\limsup_{n\to\infty} X_n = \infty$. Define

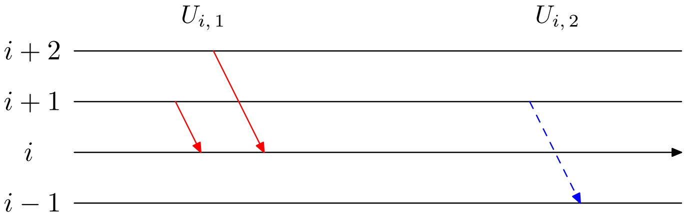
$$T_1 = \inf\{n > 0 : X_n > 0\}.$$

Let $U_0 = \mathbf{e}_1$ and for $i \le 0,\ l = 1, ..., L$,

$$U_{i,l} = \#\{0 \le n < T_1 : X_n > i, X_{n+1} = i - l + 1\}.$$

Set

$$U_i = (U_{i,1}, ..., U_{i,L}).$$



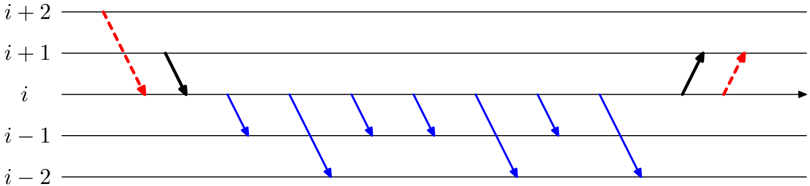


Figure. The figure illustrates the offspring born to a type-1 particle. It has 4 type-1 children and 3 type-2 children.

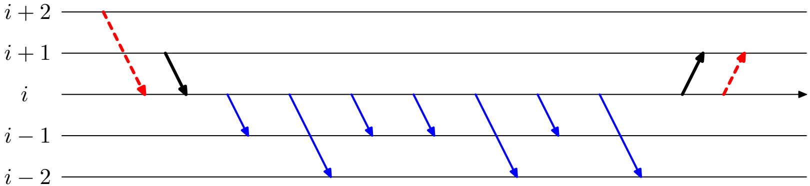


Figure. The figure illustrates the offspring born to a type-1 particle. It has 4 type-1 children and 3 type-2 children.

$$P_\omega\left(U_{i-1}=(u_1,...,u_L)\middle|U_i=\mathbf{e}_1\right)$$
$$=\frac{(u_1+...+u_L)!}{u_1!\cdots u_L!}\omega_i(-1)^{u_1}\cdots\omega_i(-L)^{u_L}\omega_i(1).$$

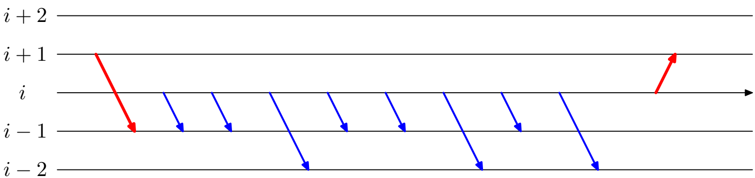


Figure. The figure illustrates the offspring born to a type-2 particle. It gives birth to a type-1 child with probability one. Then it gives births to certain particles with common distribution as type-1 particles.

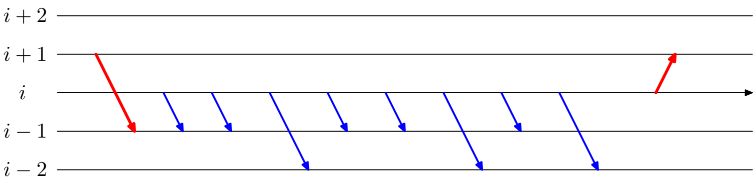


Figure. The figure illustrates the offspring born to a type-2 particle. It gives birth to a type-1 child with probability one. Then it gives births to certain particles with common distribution as type-1 particles.

For $L \geq l \geq 2,$

$$P_\omega\left(U_{i-1} = (u_1, ..., \mathbf{1} + u_{l-1}, ..., u_L) \middle| U_i = \mathbf{e}_l\right)$$
$$= \frac{(u_1 + ... + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(1).$$

Theorem(Hong and Wang 2013)

Suppose that $\limsup_{n\to\infty} X_n = \infty$. Then $U_0, U_{-1}, U_{-2}, ...$ forms an L-type branching process whose offspring distributions are

$$P_\omega(U_{i-1} = (u_1, ..., u_L)\big|U_i = e_1)$$
$$= \frac{(u_1 + ... + u_L)!}{u_1! \cdots u_L!}\omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L}\omega_i(1),$$

and for $2 \le l \le L$,

$$P_\omega\left(U_{i-1} = (u_1, ..., 1 + u_{l-1}, ..., u_L)\big|U_i = e_l\right)$$
$$= \frac{(u_1 + ... + u_L)!}{u_1! \cdots u_L!}\omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L}\omega_i(1).$$

Furthermore,

$$T_1 = 1 + \sum_{i\le 0} U_i(2, 1, ..., 1)^t.$$

Branching Structure for (2,2) RW

Consider (2,2) RW. Suppose $\limsup_{n\to\infty} X_n = \infty$. Define

$$T_1 = \inf\{n \ge 0 : X_n > 0\}.$$

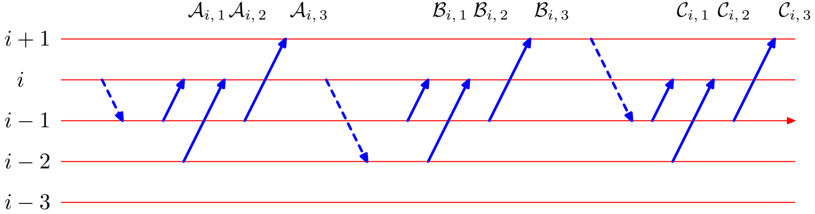


Figure. The figure illustrates type $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ excursions at i. We draw only the first step and the last step, omitting all things between these two steps. Between these two steps, the walk walks below $i-1$.

Define for $i \leq 0$ and $j = 1, 2, 3$,

$$A_{i,j} = \#\{\mathcal{A}_{i,j} \text{ excursions before } T_1\},$$

$$B_{i,j} = \#\{\mathcal{B}_{i,j} \text{ excursions before } T_1\},$$

$$C_{i,j} = \#\{\mathcal{C}_{i,j} \text{ excursions before } T_1\},$$

$$U_i = (A_{i,1}, A_{i,2}, A_{i,3}, B_{i,1}, B_{i,2}, B_{i,3}, C_{i,1}, C_{i,2}, C_{i,3}).$$

Define for $i \leq 0$ and $j = 1, 2, 3$,

$$A_{i,j} = \#\{\mathcal{A}_{i,j} \text{ excursions before } T_1\},$$

$$B_{i,j} = \#\{\mathcal{B}_{i,j} \text{ excursions before } T_1\},$$

$$C_{i,j} = \#\{\mathcal{C}_{i,j} \text{ excursions before } T_1\},$$

$$U_i = (A_{i,1}, A_{i,2}, A_{i,3}, B_{i,1}, B_{i,2}, B_{i,3}, C_{i,1}, C_{i,2}, C_{i,3}).$$

Theorem(Hong and Wang 2014)

Suppose that $\limsup_{n\to\infty} X_n = \infty$. Then $\{U_i\}_{i\leq 1}$ is a 9-type non-homogeneous branching processes whose immigration law and offsprings distributions will be stated below.

Theorem

Suppose that $\limsup_{n\to\infty} X_n = \infty$. Then

$$T_1 = 1 + \sum_{i\leq 0} U_i(2,2,1,1,1,0,2,2,1)^t,$$

$$E_\omega^0(T_1) = 1 + \sum_{i\leq 0} u_1 Q_0 \cdots Q_i(2,2,1,1,1,0,2,2,1)^t$$

where $Q_i \in \mathbb{R}^9 \times \mathbb{R}^9$, $u_1 \in \mathbb{R}^9$ depending only on ω.

Define for $k \le i$,

$$f_k(i, i+1) = P_\omega^k(\text{the walk hits } (i, \infty) \text{ at } i+1);$$

$$f_k(i, i+2) = P_\omega^k(\text{the walk hits } (i, \infty) \text{ at } i+2).$$

Define indexes $\alpha_{i,1}$, $\alpha_{i,3}$ and $\alpha_{i,2}$ correspondingly to $\mathcal{A}_{i,1}$ $\mathcal{A}_{i,3}$, and $\mathcal{A}_{i,2}$.

Define for $k \le i$,

$$f_k(i, i+1) = P_\omega^k(\text{the walk hits } (i, \infty) \text{ at } i+1);$$

$$f_k(i, i+2) = P_\omega^k(\text{the walk hits } (i, \infty) \text{ at } i+2).$$

Define indexes $\alpha_{i,1}$, $\alpha_{i,3}$ and $\alpha_{i,2}$ correspondingly to $\mathcal{A}_{i,1}$ $\mathcal{A}_{i,3}$, and $\mathcal{A}_{i,2}$. Let

$$\begin{aligned}
\alpha_{i,1} := \omega_i(-1) \sum_{n,m \ge 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1) f_{i-2}(i-2, i-1)]^n \\
\times [\omega_{i-1}(-2) f_{i-3}(i-2, i-1)]^m \omega_{i-1}(1), \\
\alpha_{i,3} := \omega_i(-1) \sum_{n,m \ge 0} \frac{(n+m)!}{n!m!} [\omega_{i-1}(-1) f_{i-2}(i-2, i-1)]^n \\
\times [\omega_{i-1}(-2) f_{i-3}(i-2, i-1)]^m \omega_{i-1}(2), \\
\alpha_{i,2} := \omega_i(-1) - \alpha_{i,1} - \alpha_{i,3}.
\end{aligned}$$

$$\alpha_{i,1} = \frac{\omega_i(-1)\omega_{i-1}(1)}{1 - \omega_{i-1}(-1) f_{i-2}(i-2, i-1) - \omega_{i-1}(-2) f_{i-3}(i-2, i-1)},$$
$$\alpha_{i,3} = \frac{\omega_i(-1)\omega_{i-1}(2)}{1 - \omega_{i-1}(-1) f_{i-2}(i-2, i-1) - \omega_{i-1}(-2) f_{i-3}(i-2, i-1)},$$
$$\alpha_{i,2} := \omega_i(-1) - \alpha_{i,1} - \alpha_{i,3}.$$

Define similarly $\beta_{i,1},\ \beta_{i,2},\ \beta_{i,3},\ \gamma_{i,1},\ \gamma_{i,2},\ \gamma_{i,3}$.

$$\alpha_{i,1} = \frac{\omega_i(-1)\omega_{i-1}(1)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)},$$
$$\alpha_{i,3} = \frac{\omega_i(-1)\omega_{i-1}(2)}{1 - \omega_{i-1}(-1)f_{i-2}(i-2, i-1) - \omega_{i-1}(-2)f_{i-3}(i-2, i-1)},$$
$$\alpha_{i,2} := \omega_i(-1) - \alpha_{i,1} - \alpha_{i,3}.$$

Define similarly $\beta_{i,1},\ \beta_{i,2},\ \beta_{i,3},\ \gamma_{i,1},\ \gamma_{i,2},\ \gamma_{i,3}.$

$$\mathcal{A}_{i,1} \leftrightarrow \alpha_{i,1},\ \mathcal{A}_{i,2} \leftrightarrow \alpha_{i,2},\ \mathcal{A}_{i,3} \leftrightarrow \alpha_{i,3};$$
$$\mathcal{B}_{i,1} \leftrightarrow \beta_{i,1},\ \mathcal{B}_{i,2} \leftrightarrow \beta_{i,2},\ \mathcal{B}_{i,3} \leftrightarrow \beta_{i,3};$$
$$\mathcal{C}_{i,1} \leftrightarrow \gamma_{i,1},\ \mathcal{C}_{i,2} \leftrightarrow \gamma_{i,2},\ \mathcal{C}_{i,3} \leftrightarrow \gamma_{i,3}.$$

Immigration Law

Figure. Adding the imaginary step $\{1 \to 0\}$, the path before T_1 forms a type $\mathcal{A}$ particle, it may be a $\mathcal{A}_{1,1}$, $\mathcal{A}_{1,2}$ or $\mathcal{A}_{1,3}$ excursion.

$$
\begin{aligned}
P_\omega(U_1 = \mathbf{e}_1) = P_\omega^0(A_{1,1} = 1) &= \frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}}, \\
P_\omega(U_1 = \mathbf{e}_3) = P_\omega^0(A_{1,3} = 1) &= \frac{\alpha_{1,3}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}}, \qquad (1) \\
P_\omega(U_1 = \mathbf{e}_2) = P_\omega^0(A_{1,2} = 1) &= \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}}.
\end{aligned}
$$

(a) Offspring distributions of $\mathcal{A}_{i+1,1}$, $\mathcal{A}_{i+1,3}$, $\mathcal{C}_{i+1,1}$ and $\mathcal{C}_{i+1,3}$ particles

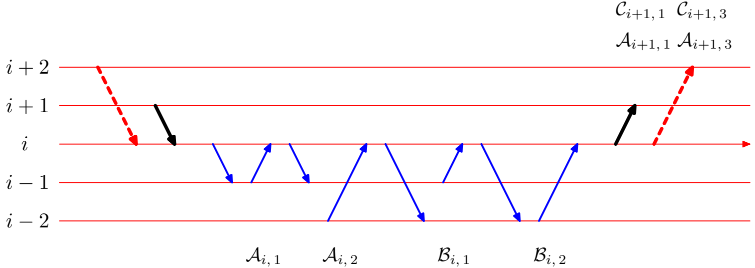


Figure. $\mathcal{A}_{i+1,\,1}$, $\mathcal{A}_{i+1,\,3}$, $\mathcal{C}_{i+1,\,1}$ and $\mathcal{C}_{i+1,\,3}$ share the same offspring distribution. They could only give births to $\mathcal{A}_{i,\,1}$, $\mathcal{A}_{i,\,2}$, $\mathcal{B}_{i,\,1}$ and $\mathcal{B}_{i,\,2}$ particles.

(a) Offspring distributions of $\mathcal{A}_{i+1,1}$, $\mathcal{A}_{i+1,3}$, $\mathcal{C}_{i+1,1}$ and $\mathcal{C}_{i+1,3}$ particles

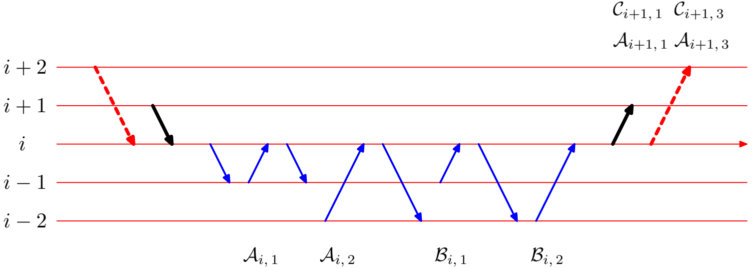


Figure. $\mathcal{A}_{i+1,\,1}$, $\mathcal{A}_{i+1,\,3}$, $\mathcal{C}_{i+1,\,1}$ and $\mathcal{C}_{i+1,\,3}$ share the same offspring distribution. They could only give births to $\mathcal{A}_{i,\,1}$, $\mathcal{A}_{i,\,2}$, $\mathcal{B}_{i,\,1}$ and $\mathcal{B}_{i,\,2}$ particles.

With $\zeta_i = 1 - \alpha_{i,1} - \alpha_{i,2} - \beta_{i,1} - \beta_{i,2}$, for $k = 1, 3, 7, 9$,

$$P_\omega^0\big(U_i = (a, b, 0, c, d, 0, 0, 0, 0)\big|U_{i+1} = \mathbf{e}_k\big)$$
$$= \frac{(a+b+c+d)!}{a!b!c!d!}\alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d \zeta_i.$$

(b) Offspring distributions of $\mathcal{A}_{i+1,2}$, and $\mathcal{C}_{i+1,2}$ particles

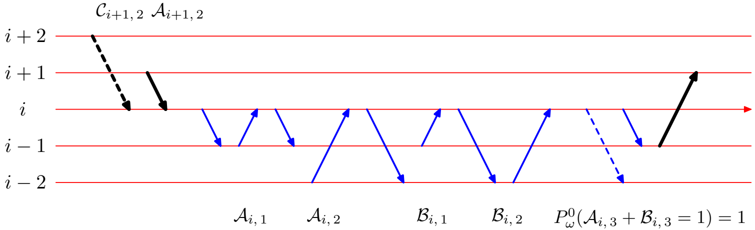


Figure. Offsprings of $\mathcal{A}_{i+1,\,2}$, $\mathcal{C}_{i+1,\,2}$. Before the last step happens, with probability 1, a $\mathcal{B}_{i,\,3}$ or $\mathcal{A}_{i,\,3}$ excursion would be born.

$$P^0_\omega(A_{i,3}=1\big|U_{i+1}=\mathbf{e}_2 \text{ or } \mathbf{e}_8)=\frac{\alpha_{i,3}}{\alpha_{i,3}+\beta_{i,3}},$$

$$P^0_\omega(B_{i,3}=1\big|U_{i+1}=\mathbf{e}_2 \text{ or } \mathbf{e}_8)=\frac{\beta_{i,3}}{\alpha_{i,3}+\beta_{i,3}}.$$

$$
\begin{aligned}
P_\omega^0\Big(U_i = (a, b, \mathbf{1}, c, d, 0, 0, 0, 0)\Big| U_{i+1} = \mathbf{e}_2 \text{ or } \mathbf{e}_8\Big) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d \zeta_i \frac{\alpha_{i,3}}{\alpha_{i,3} + \beta_{i,3}},
\end{aligned}
$$

$$
\begin{aligned}
P_\omega^0\Big(U_i = (a, b, 0, c, d, \mathbf{1}, 0, 0, 0)\Big| U_{i+1} = \mathbf{e}_2 \text{ or } \mathbf{e}_8\Big) \\
= \frac{(a+b+c+d)!}{a!b!c!d!} \alpha_{i,1}^a \alpha_{i,2}^b \beta_{i,1}^c \beta_{i,2}^d \zeta_i \frac{\beta_{i,3}}{\alpha_{i,3} + \beta_{i,3}}.
\end{aligned}
$$

(c) Offspring distributions of $\mathcal{B}_{i+1,1}$, and $\mathcal{B}_{i+1,3}$ particles

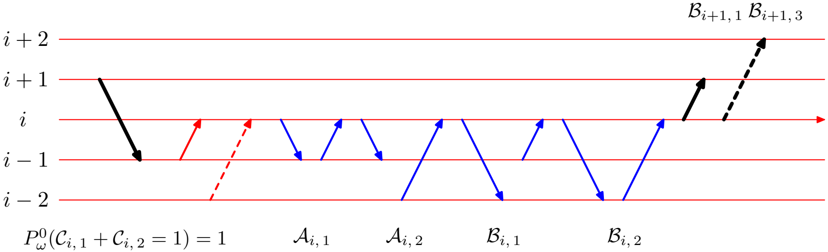


Figure. The offsprings of $\mathcal{B}_{i+1,1}$ and $\mathcal{B}_{i+1,3}$. Since at last, the walk jumps from i to some position above i, before the last step happens, it must return to i from below. Therefore with probability 1, a $\mathcal{C}_{i,1}$ or a $\mathcal{C}_{i,2}$ excursion would be born.

(d) Offspring distribution of $\mathcal{B}_{i+1,2}$ particles

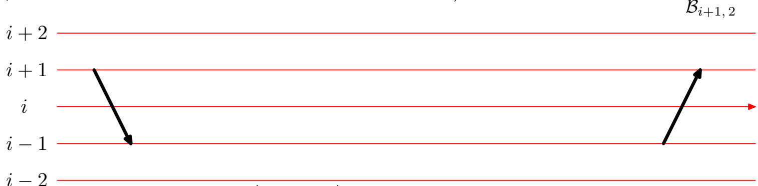


Figure. The offsprings (**case** 1) of $\mathcal{B}_{i+1,\,2}$ excursion. **The walk never visited i between the first and the last step**. Therefore, only a type $\mathcal{C}_{i,\,3}$ excursion would be born.

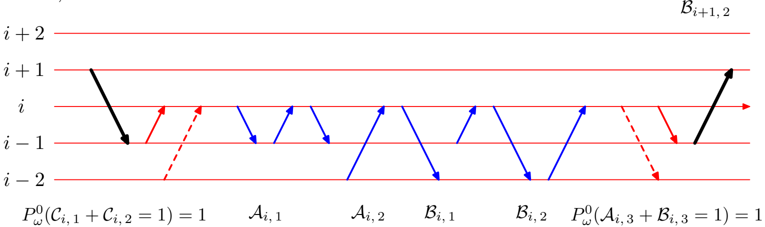


Figure. The offsprings (**case** 2) of $\mathcal{B}_{i+1,\,2}$ excursion. **Between the first and the last step, the walk did visit i.** Therefore $P^0_\omega(\mathcal{C}_{i,\,1}+\mathcal{C}_{i,\,2}=1)=1$ and $P^0_\omega(\mathcal{A}_{i,\,3}+\mathcal{B}_{i,\,3}=1)=1$.

$$P^0_\omega(U_i = (0, ..., 0, 1)\big|U_{i+1} = \mathbf{e}_5) = \frac{\gamma_{i,3}}{\beta_{i+1,2}},$$

$$P^0_\omega(U_i = (0, ..., 0, 1)\big|U_{i+1} = \mathbf{e}_5) = \frac{\gamma_{i,3}}{\beta_{i+1,2}},$$

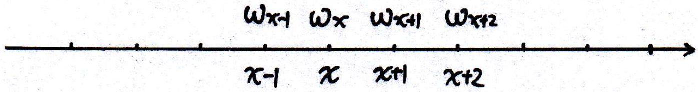
$$P^0_\omega(U_i = (a, b, \mathbf{1}, c, d, 0, \mathbf{1}, 0, 0)\big|U_{i+1} = \mathbf{e}_5)$$
$$= \frac{(a+b+c+d)!}{a!b!c!d!}\alpha^a_{i,1}\alpha^b_{i,2}\beta^c_{i,1}\beta^d_{i,2}\zeta_i\frac{(\beta_{i+1,2}-\gamma_{i,3})\gamma_{i,1}\alpha_{i,3}}{\beta_{i+1,2}(\gamma_{i,1}+\gamma_{i,2})(\alpha_{i,3}+\beta_{i,3})},$$

$$P^0_\omega(U_i = (a, b, \mathbf{1}, c, d, 0, 0, \mathbf{1}, 0)\big|U_{i+1} = \mathbf{e}_5)$$
$$= \frac{(a+b+c+d)!}{a!b!c!d!}\alpha^a_{i,1}\alpha^b_{i,2}\beta^c_{i,1}\beta^d_{i,2}\zeta_i\frac{(\beta_{i+1,2}-\gamma_{i,3})\gamma_{i,2}\alpha_{i,3}}{\beta_{i+1,2}(\gamma_{i,1}+\gamma_{i,2})(\alpha_{i,3}+\beta_{i,3})},$$

$$P^0_\omega(U_i = (a, b, 0, c, d, \mathbf{1}, \mathbf{1}, 0, 0)\big|U_{i+1} = \mathbf{e}_5)$$
$$= \frac{(a+b+c+d)!}{a!b!c!d!}\alpha^a_{i,1}\alpha^b_{i,2}\beta^c_{i,1}\beta^d_{i,2}\zeta_i\frac{(\beta_{i+1,2}-\gamma_{i,3})\gamma_{i,1}\beta_{i,3}}{\beta_{i+1,2}(\gamma_{i,1}+\gamma_{i,2})(\alpha_{i,3}+\beta_{i,3})},$$

$$P^0_\omega(U_i = (a, b, 0, c, d, \mathbf{1}, 0, \mathbf{1}, 0)\big|U_{i+1} = \mathbf{e}_5)$$
$$= \frac{(a+b+c+d)!}{a!b!c!d!}\alpha^a_{i,1}\alpha^b_{i,2}\beta^c_{i,1}\beta^d_{i,2}\zeta_i\frac{(\beta_{i+1,2}-\gamma_{i,3})\gamma_{i,2}\beta_{i,3}}{\beta_{i+1,2}(\gamma_{i,1}+\gamma_{i,2})(\alpha_{i,3}+\beta_{i,3})}.$$

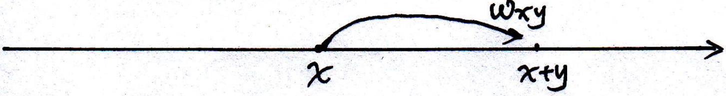
Random walk in random environment



$\omega_x = (\omega_{xy})_{y\in\mathbb{Z}}$, a prob. meas. on $\mathbb{Z}$

$\omega = (\cdots, \omega_{x-1}, \omega_x, \omega_{x+1}, \omega_{x+2}, \dots)$

$\mathbb{P}$ is a prob. meas. making ω an i.i.d. or ergodic sequence.



$P_\omega(S_{n+1} = x+y \mid S_n = x) = \omega_{xy}$

Ω : collection of $\omega = (\omega_x)_{x\in\mathbb{Z}}$ where for $x \in \mathbb{Z}$, $\omega_x = (\omega_{xy})_{y\in\mathbb{Z}}$ is a probability measure on $\mathbb{Z}$.

θ : shift operator on Ω defined by $(\theta\omega)_x := \omega_{x+1}$.

$\mathcal{F}$: Borel σ-algebra on Ω.

$\mathbb{P}$: a probability measure on $(\Omega, \mathcal{F})$, being i.i.d. or ergodic.

For a realization of ω, consider a Markov chain $\{S_n\}_{n\geq 0}$ with transitional probabilities

$$P_\omega^{x_0}(S_{n+1} = x+y \big| S_n = x) = \omega_{xy} \text{ for all } n \geq 0,\ P_\omega^{x_0}(S_0 = x_0) = 1.$$

$\{S_n\}$: random walk (**with unbounded jumps**) in random environment ω.

$P_\omega^{x_0}$: the quenched law.

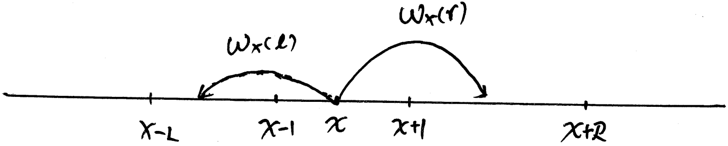
$P^{x_0}(\cdot) = \int P_\omega^{x_0}(\cdot)\mathbb{P}(d\omega)$: the annealed law.

Some applications of Branching Structure for random walk

If for some $1 \le L, R \in \mathbb{Z}$,

$$\mathbb{P}(\omega_{0y} = 0, \text{ for } y < -L \text{ and } y > R) = 1,$$

$\{S_n\}$ is called RWRE with bounded jumps (**(L,R) RWRE**).

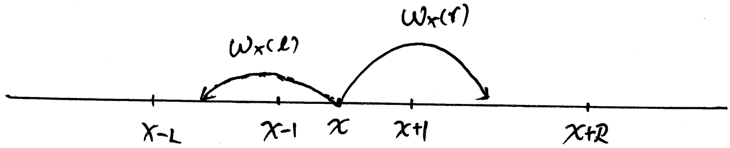


Some applications of Branching Structure for random walk

If for some $1 \le L, R \in \mathbb{Z}$,

$$\mathbb{P}(\omega_{0y} = 0,\ \text{for } y < -L \text{ and } y > R) = 1,$$

$\{S_n\}$ is called RWRE with bounded jumps (**(L,R) RWRE**).



Wang (2013) prove the **stable law for (L,1) RWRE**, partially generalizing Kesten, Kozlov and Spitzer (1975).

Bremont (2009) proved a LLN for (L, R) RWRE, but for $\min\{L, R\} \geq 2$, **no explicit velocity was given**.

Bremont (2009) proved a LLN for (L, R) RWRE, but for $\min\{L, R\} \geq 2$, **no explicit velocity was given**.
Wang and Hong (2014) reproved the LLN and **gave explicitly the velocity** by using the branching structure of (L, R) RWRE.

Bremont (2009) proved a LLN for (L, R) RWRE, but for $\min\{L, R\} \geq 2$, **no explicit velocity was given**.
Wang and Hong (2014) reproved the LLN and **gave explicitly the velocity** by using the branching structure of (L, R) RWRE.

Hong, Zhou and Zhao (2014) gave the explicit stationary distribution of the (L, 1)-reflecting random walk by using the (L, 1) branching structure.

Bremont (2009) proved a LLN for (L, R) RWRE, but for $\min\{L, R\} \geq 2$, **no explicit velocity was given**.
Wang and Hong (2014) reproved the LLN and **gave explicitly the velocity** by using the branching structure of (L, R) RWRE.

Hong, Zhou and Zhao (2014) gave the explicit stationary distribution of the (L, 1)-reflecting random walk by using the (L, 1) branching structure.

Hong and Shun (2013) proved the renewal theorem for (L, 1) RWRE.

Bremont (2009) proved a LLN for (L, R) RWRE, but for $\min\{L, R\} \geq 2$, **no explicit velocity was given**.
Wang and Hong (2014) reproved the LLN and **gave explicitly the velocity** by using the branching structure of (L, R) RWRE.

Hong, Zhou and Zhao (2014) gave the explicit stationary distribution of the (L, 1)-reflecting random walk by using the (L, 1) branching structure.

Hong and Shun (2013) proved the renewal theorem for (L, 1) RWRE.

Wang (2014^+) proved the LLN for BDP with bounded jumps in random environment with explicit velocity.

Bremont (2009) proved a LLN for (L, R) RWRE, but for $\min\{L, R\} \geq 2$, **no explicit velocity was given**.
Wang and Hong (2014) reproved the LLN and **gave explicitly the velocity** by using the branching structure of (L, R) RWRE.

Hong, Zhou and Zhao (2014) gave the explicit stationary distribution of the (L, 1)-reflecting random walk by using the (L, 1) branching structure.

Hong and Shun (2013) proved the renewal theorem for (L, 1) RWRE.

Wang (2014^+) proved the LLN for BDP with bounded jumps in random environment with explicit velocity.

Hong and Yang (2014^+) showed the convergence of the local time of (1, L) RW to Brownian local time.

RWRE with unbounded jumps

Up to our knowledge, there are only 3 papers, Andjel (1988), Comets and Popov (2012), Wang (2014^+).

RWRE with unbounded jumps

Up to our knowledge, there are only 3 papers, Andjel (1988), Comets and Popov (2012), Wang (2014^+).

Condition A

(A1) $\mathbb{P}$ is stationary and ergodic;

(A2) $\mathbb{P}$-a.s. $\sum_{n=0}^{\infty} P_\omega(S_n = y | S_0 = x) > 0$, for all $x, y \in \mathbb{Z}$;

(A3) There exists some $C > 0$ such that for all $s > 0$, $\mathbb{P}$-a.s.

$$\sum_{|j| \geq s} \omega_{0j} < Ce^{-s}.$$

Andjel 1988, 0-1 law

Under condition (A), we have $P(X_n \to \infty) = 1$ or $P(X_n \to -\infty) = 1$ or $P(X_n \text{ is recurrent}) = 1$.

For $\varrho \in \mathbb{N}$, let

$$\omega^{\varrho}_{xy} = \begin{cases} \omega_{xy}, & \text{if } 0 < |y| < \varrho, \\ 0, & \text{if } |y| \geq \varrho, \\ \omega_{x0} + \sum_{y:|y|\geq\rho} \omega_{xy}, & \text{if } y = 0. \end{cases}$$

Let $\{S^{\varrho}_n\}$ be random walk in truncated environment ω^{ϱ}. Define

$$N^{\varrho}_{\infty}(x) = \sum_{n=0}^{\infty} 1_{\{S^{\varrho}_n = x\}}.$$

Condition B

(B1) $\mathbb{P}$ is stationary and ergodic;

(B2) $\mathbb{P}(\omega_{01} > \epsilon) = 1$ for some $\epsilon > 0$;

(B3) $\exists r > 0, \alpha > 1$ such that for all $s \geq 1$, $\sum_{|y|>s} \omega_{0y} \leq rs^{-\alpha}$;

(B4) $\exists$ non-increasing $g \geq 0$ such that $\sum_{k=1}^{\infty} kg(k) < \infty$ and a finite $\varrho_0 > 0$ such that for all $x \leq 0$ and $\varrho > \varrho_0$

$$\mathbb{P}\text{-a.s., } E^0_{\omega}(N^{\varrho}_{\infty}(x)) \leq g(|x|).$$

("The strong uniform transience to the right", it precludes the existence of trap.)

Theorem(Comets and Popov 2012, LLN)

Suppose that Condition (B) is satisfied. Then for all $\varrho > \varrho_0$, $\exists\ v_\varrho > 0$ such that

$$P\text{-a.s. } \frac{S_n^\rho}{n} \to v_\varrho;$$

moreover, there exists $\mathbb{Q}^\varrho \ll \mathbb{P}$ such that $v_\varrho = \int_\Omega E_\omega^0(S_1^\varrho)d\mathbb{Q}^\varrho$ and $v_\varrho \to v_\infty$ as $\varrho \to \infty$.

Theorem(Comets and Popov 2012, LLN)

Suppose that Condition (B) is satisfied. Then for all $\varrho > \varrho_0$, $\exists\ v_\varrho > 0$ such that

$$P\text{-a.s. } \frac{S_n^\rho}{n} \to v_\varrho;$$

moreover, there exists $\mathbb{Q}^\varrho \ll \mathbb{P}$ such that $v_\varrho = \int_\Omega E_\omega^0(S_1^\varrho)d\mathbb{Q}^\varrho$ and $v_\varrho \to v_\infty$ as $\varrho \to \infty$.

Remark

(B3) requires the uninform and integrable polynomial tail of the jumps. The authors said "It is a challenging problem to find weaker conditions that still permit to obtain LLN for RWREs with unbounded jumps **with only polynomial tails.**"

Theorem(Comets and Popov 2012, LLN)

Suppose that Condition (B) is satisfied. Then for all $\varrho > \varrho_0$, $\exists\ v_\varrho > 0$ such that

$$P\text{-a.s. } \frac{S_n^\rho}{n} \to v_\varrho;$$

moreover, there exists $\mathbb{Q}^\varrho \ll \mathbb{P}$ such that $v_\varrho = \int_\Omega E_\omega^0(S_1^\varrho)d\mathbb{Q}^\varrho$ and $v_\varrho \to v_\infty$ as $\varrho \to \infty$.

Remark

(B3) requires the uninform and integrable polynomial tail of the jumps. The authors said "It is a challenging problem to find weaker conditions that still permit to obtain LLN for RWREs with unbounded jumps **with only polynomial tails.**"
(B4) is also called "the strong uniform transience to the right", it precludes the existence of trap. The authors said "In particular, **it would be especially interesting to substitute the current condition by a weaker one**; however, at the moment we do not have any concrete results and/or plausible conjectures which go in that direction."

Condition C

(C1) $\mathbb{P}$ is stationary and ergodic.

(C2) There exists $\varepsilon > 0$ such that $\mathbb{P}(\omega_{01} > \varepsilon) = 1$.

(C3) There exist small $\varepsilon_0 > 0$ and proper $D > 0$, such that $\mathbb{P}$-a.s.,

$$\omega_{0j} < D|j|^{-(3+\varepsilon_0)}.$$

Condition C

(C1) $\mathbb{P}$ is stationary and ergodic.

(C2) There exists $\varepsilon > 0$ such that $\mathbb{P}(\omega_{01} > \varepsilon) = 1$.

(C3) There exist small $\varepsilon_0 > 0$ and proper $D > 0$, such that $\mathbb{P}$-a.s.,

$$\omega_{0j} < D|j|^{-(3+\varepsilon_0)}.$$

Define $T = \inf\{n > 0 : S_n > 0\}$, $U_k = \#\{0 \le n < T : S_n = k\}$.

Theorem(Wang 2014)

Suppose that Condition C holds and $E(T) < \infty$. Then

$$P\text{-a.s.,} \quad \lim_{n\to\infty} \frac{S_n}{n} = v_{\mathbb{P}} > 0$$

where

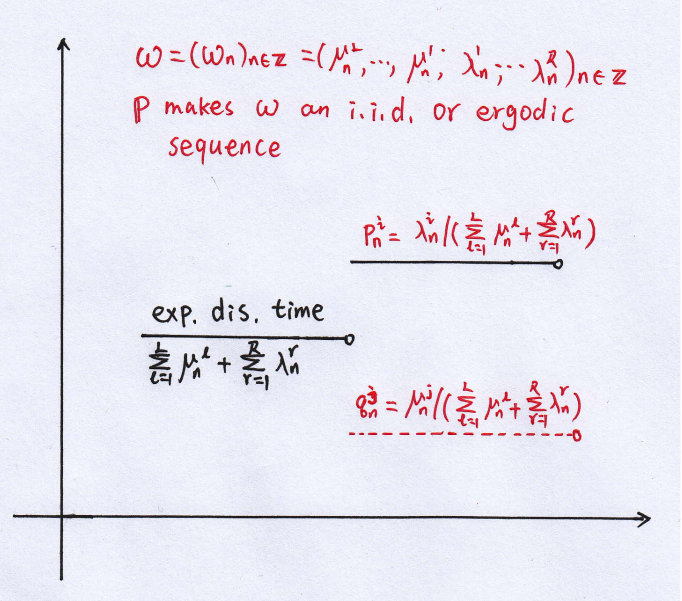
$$v_{\mathbb{P}} = \frac{\mathbb{E}\left(\sum_{i=1}^{\infty} \sum_{k\le 0} E_{\theta^{-k}\omega}\left(U_k | S_{T=i}\right) \sum_{j\in\mathbb{Z}} j\omega_{0j}\right)}{\sum_{i=1}^{\infty} E\left(T | S_T = i\right)}.$$

Remark

(1) We prove the LLN when the tails of the jumps decay polynomially.
(2) We do not need the uniform transience condition (B4) used in Comets and Popov 2012. However
(3) We don't know how to calculate $E^0_\omega(T_1)$, so we could not give explicitly the velocity $v_{\mathbb{P}}$, except for some special case e.g.

$$\mathbb{P}(\omega_{0y} = 0 \text{ for all } y \geq 2) = 1.$$

BDP with bounded jumps in random environment



Space of Random Environment

$L, \ R \geq 1$ are two integers(**jump size**).

Ω : collection of $\omega = (\omega_i)_{i\in\mathbb{Z}} = (\mu_i^L, ..., \mu_i^1, \lambda_i^1, ..., \lambda_i^R)_{i\in\mathbb{Z}}$, $\mu_i^l, \lambda_i^r \geq 0$, $i \in \mathbb{Z}$, $l = 1, .., L$, $r = 1, ..., R$.

$\mathcal{F}$: Borel σ-algebra on Ω.

θ : shift operator on Ω defined by $(\theta\omega)_i = \omega_{i+1}$.

$\mathbb{P}$: a probability measure on $(\Omega, \mathcal{F})$ which is assumed to be i.i.d. or sometimes stationary and ergodic.

Random environment ω is a random element of Ω chosen according to $\mathbb{P}$.

(L,R) BDPRE

Given a realization of ω,
let $\{N_t\}_{t\geq 0}$ be a continuous time Markov chain,
which waits at a state n an exponentially distributed time with parameter $\sum_{l=1}^{L}\mu_n^l+\sum_{r=1}^{R}\lambda_n^r$ and then
jumps to $n-i$ with probability $\mu_n^i/(\sum_{l=1}^{L}\mu_n^l+\sum_{r=1}^{R}\lambda_n^r)$, $i=1,...,L$
or to $n+j$ with probability $\lambda_n^j/(\sum_{l=1}^{L}\mu_n^l+\sum_{r=1}^{R}\lambda_n^r)$, $j=1,...,R$.
$\{N_t\}_{t\geq 0}$ is called a *birth and death process with bounded jumps in random environment* ((L,R) BDPRE in short).

P_ω : **quenched probability;**

P : **annealed probability.**

Condition D

(D1) $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ forms a stationary and ergodic system.

(D2) the measure $\mathbb{P}$ is uniformly elliptic, that is,

$$\mathbb{P}\Big(\varepsilon < \mu_0^l, \lambda_0^r < M, 1 \le l \le L, 1 \le r \le R\Big) = 1$$

for some small $\varepsilon > 0$ and large $M > 0$.

Define $T_1 = \inf[t > 0 : N_t > 0]$.

Theorem (LLN for (L,R) BDPRE)

Suppose that conditions (D) holds and $\gamma_R \geq 0$. Then
(a) $ET_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = v_{\mathbb{P}} > 0$, P-a.s.;
(b) $ET_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

$v_{\mathbb{P}} =$

$$\frac{\mathbb{E}\Big(\sum_{r=1}^{R} \sum_{k\leq 0} E_{\theta^{-k}\omega}\big(\sum_{j=1}^{U_k} \xi_{kj} | N_{T_1} = r\big)\big(\sum_{l=1}^{L} (-l)\mu_0^l + \sum_{r=1}^{R} r\lambda_0^r\big)\Big)}{\sum_{r=1}^{R} E(T_1 | N_{T_1} = r)}$$

$U_k := \#\{n : N_{\tau_n} = k, \tau_n < T_1\}$,

$P_\omega(\xi_{kj} > t) = e^{-(\sum_{l=1}^{L} \mu_k^l + \sum_{r=1}^{R} \lambda_k^r)t}, \ t \geq 0.$

Theorem (LLN for (2,2) BDPRE)

Let $\pi(\omega)$ and $D(\omega)$ be certain functions of ω. Suppose $L = R = 2$ and $\gamma_R \geq 0$. Then $\mathbb{P}$-a.s.,

(a) $\mathbb{E}(\pi(\omega)) < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = \frac{\mathbb{E}\left(\pi(\omega)(2\lambda_0^2+\lambda_0^1-\mu_0^1-2\mu_0^2)\right)}{\mathbb{E}(D(\omega))}$;

(b) $\mathbb{E}(\pi(\omega)) = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$.

Idea: Let $\{X_n\} = \{N_{nh}\}$ be h-skeleton process of $\{N_t\}$.

$\{X_n\}$ is a RWRE with unbounded jumps.

LLN of $\{X_n\} \Rightarrow$ LLN of $\{N_t\}$.

Using the branching structure for (L, R) random walk, we calculate

$$E_{\theta^{-k}\omega}\Big(\sum_{j=1}^{U_k} \xi_{kj} | N_{T_1} = r\Big) \text{ and } \sum_{r=1}^{R} E_\omega(T_1 | N_{T_1} = r),$$

which lead to the **explicit velocity** of LLN of $\{N_t\}$.

Reference

[AF14] Afanasyev, V.I., Conditional limit theorem for the maximum of random walk in random environment, Theory Probab. Appl. Vol. 58(4), pp 525-545

[AL99] Alili, S., Asymptotic behavior for random walks in random environments, J. Appl. Probab., Vol. 36, pp 334-349, 1999

[AD88] Andjel, E.D., A zero or one law for one dimensional random walks in random environments, Ann. Probab., Vol. 16(2), pp 722-729, 1988

[BR09] Brémont, J., One-dimensional finite range random walk in random medium and invariant measure equation, Ann. Inst. H. PoincaréProbab. Statist., Vol. 45, pp 70-103, 2009

[CP12] Comets, F., Popov, S., Ballistic regime for random walks in random environment with unbounded jumps and Knudsen billiards, Ann. Inst. H. PoincaréProbab., Statist., Vol. 48(3), pp 721-744, 2012

[DW75] Dwass, M., Branching processes in simple random walk. Proc. Amer. Math. Soc. Vol. 51(2), pp 270-274, 1975

[GS02] Gantert, N., Shi, Z., Many visits to a single site by a transient random walk in random environment, Stoc. Proc. Their Appl. Vol. 99, pp 159-176, 2002

[HA52] Harris, T. E., First Passage and Recurrence Distributions, Trans. Amer. Math. Soc. Vol. 73(3), pp 471-486, 1952

[HS13] Hong, W.M., Sun, H.Y., Renewal theorem for (L; 1)-random walk in random environment, Acta Math. Sin. (Engl. Ser.), Vol. 33(6), pp 1736-1748, 2013

[HY15] Hong, W.M., Yang H., Scaling limit of the local time of the (1, L)-random walk, Submitted, 2014

[HZ14] Hong, W.M., Zhou, K., and Yi Qiang Q. Zhao, Explicit stationary distribution of the (L, 1)-reflecting random walk on the half line, Acta Math. Sin. (Engl. Ser.), Vol. 30(3), pp 371-388, 2014

[KE84] Key, E. S., Recurrence and transience criteria for random walk in a random environment, Ann. Probab., Vol. 12, pp 529-560, 1984

Reference

[RO84] Rogers, L.C.G., Brownian local times and branching processes, Séminaire de Probabilités XVIII 1982/83; LNM Vol. 1059, pp 42-55, 1984

[WA13] Wang, H.M., A note on multitype branching process with bounded immigration in random environment, Acta Math. Sin. (Engl. Ser.), Vol. 29(6), pp 1095-1110, 2013

[WA14] Wang, H. M., Law of large numbers for random walk with unbounded jumps and BDP with bounded jumps in random environment, arXiv:1406.6222, 2014

[ZE04] Zeitouni, O., Random walks in random environment, Lecture Notes in Math. 1837, J. Picard (Ed.), pp 189-312, Springer-Verlag Berlin Heidelberg, 2004

Acknowledgement

Muchas gracias

Thanks a lot

非常感谢

hmking@mail.ahnu.edu.cn