Genealogical tree for continuous branching process

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Branching processes and their applications
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Outline

In collaboration with Y.-T. Chen (Ann. Prob. 2012) and with H. Bi (ArXiv 2013).

The model: Feller diffusion

- Random size population with indiv. of infinitesimal mass (CSBP).
- Branching population: extinction or increasing.
- Biology: stationary population.
- Consider sub-critical CSBP conditioned to survive + quadratic case $(\theta > 0)$, $t \in \mathbb{R}$:

  $$dZ_t = \sqrt{2Z_t}\, dB_t + 2(1 - \theta Z_t)\, dt.$$

  This is Feller diffusion. We have:

  $$\mathbb{E}[Z_t] = \frac{1}{\theta} \cdot$$

- $Z = (Z_t, t \in \mathbb{R})$ corresponds to an **immortal individual** with infinite birth rate or CSBP with infinite rate **immigration**.

Bottleneck effect

Results from Chen-D. (2012).

- Representation of Z:

$$Z_t = \sum_{t_i < t} Y^i_{t-t_i},$$

with $\sum_i \delta_{t_i, Y^i}(dt, dY)$ is a Poisson Point Measure with intensity:

$$2dt\, \mathbb{N}[dY],$$

and $\mathbb{N}$ is excursion measure of Feller diffusion.

- Time to the Most Recent Common Ancestor (TMRCA) of pop. living at time t:

$$A_t = \inf\{t_i,\ Y^i_{t-t_i} > 0\}.$$

- Bottleneck effect:

$$\mathbb{E}[Z_{A_t}] = \frac{2}{3}\mathbb{E}[Z_t].$$

Number of ancestors in Y

- Excursion of:

$$B_v^\theta = \sqrt{2} B'_v - 2\theta v.$$

- Ray-Knight theorem under $\mathbb{N}$:

$$(L_\sigma^t, t \geq 0) \stackrel{(d)}{=} (Y_t, t \geq 0).$$

- $M_{t-r}^t(Y)$:
  - Number of ancestors at time $t - r$ of the population living at time t,
  - Number of excursions above level $t - r$ reaching level t.
- Conditionally on Y_{t-r}

$$M_{t-r}^t(Y) \sim \mathcal{P}(c(r) Y_{t-r}),$$

with

$$c(r) = \mathbb{N}[\zeta > r] \sim_{0+} \frac{1}{r} \cdot$$

Number of ancestors in Z

- Recall:

$$Z_t = \sum_{t_i < t} Y^i_{t-t_i},$$

- Set:

$$M^t_{t-r} = \sum_{t_i < t-r} M^{t-t_i}_{t-t_i-r}(Y^i).$$

- Conditionally on Z_{t-r}:

$$M^t_{t-r} \sim \mathcal{P}(c(r) Z_{t-r}).$$

- As r goes down to 0, and since Z is continuous:

$$M^t_{t-r} \sim c(r) Z_{t-r} \sim c(r) Z_t.$$

- M^t_{t-r} finite ("Coming down from infinity") but blows up as $r \downarrow 0$.

Length of the genealogical tree

Results from Bi-D. (2015).

- For $\varepsilon > 0$, set:

$$L_t^\varepsilon = \int_\varepsilon^\infty M_{t-r}^t \, dr.$$

- Recall $M_{t-r}^t \sim c(r) Z_t$.
- Compensated length:

$$\mathcal{L}_t^\varepsilon = \int_\varepsilon^\infty M_{t-r}^t \, dr - Z_t \int_\varepsilon^\infty c(r) dr.$$

- Main result:

$$\mathcal{L}_t^\varepsilon \xrightarrow[\varepsilon \to 0]{L^2 \text{ and p.s.}} \mathcal{L}_t.$$

  and $(\mathcal{L}_t, t \in \mathbb{R})$ is (non-Markov *a priori*) càdlàg process with jumps.
- See also the talk of Götz and Pfaffelhuber-Wakolbinger-Weisshaupt (2011) for Kingman coalescent.

Law of the $\mathcal{L}_t$

- We have:

$$M^t_{t-r} \overset{(d)}{=} \sum_j \mathbf{1}_{\{\zeta_j > r\}},$$

with $\sum_j \delta_{\zeta_j}$ a Poisson Point Measure with intensity $Z_t \mathbb{N}[d\zeta]$.

- This gives $\mathcal{L}^\varepsilon_t$ is distributed as:

$$\int_\varepsilon^\infty \sum_j \mathbf{1}_{\{\zeta_j > r\}}\, dr - Z_t \int_\varepsilon^\infty \mathbb{N}[\zeta > r]\, dr.$$

That is:

$$\mathcal{L}^\varepsilon_t \overset{(d)}{=} \sum_j (\zeta_j - \varepsilon)_+ - Z_t\, \mathbb{N}\left[(\zeta_j - \varepsilon)_+\right].$$

- Deduce that:

$$\mathbb{E}\left[\mathrm{e}^{-\lambda \mathcal{L}_t} | Z_t\right] = \exp\left\{-Z_t \mathbb{N}\left[1 - \mathrm{e}^{-\lambda\zeta} - \lambda\zeta\right]\right\}.$$

and (for $\theta = 1/2$): $\quad \mathbb{N}\left[1 - \mathrm{e}^{-\lambda\zeta} - \lambda\zeta\right] = -\lambda \int_0^1 \frac{1 - v^\lambda}{1 - v}\, dv.$