Point estimation in multivariate power series offspring distiributions

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Content

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2. Family tree reconstruction
3. Estimation of the parameters.
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5. EM approach.
6. Conclusions.

The process

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$$p_{ij}^k = P(\xi_k(t, l) = (i, j)) = P\left(\xi_k^1(t, l) = i, \xi_k^2(t, l) = j\right)$$

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$$Z_k\left(t,(i,j)\right) = \sum_{l=1}^{Z_1(t)} \sum_{h=1}^{Z_2(t)} I\left(\xi_k^1(t,l) = i, \xi_k^2(t,h) = j\right)$$

is the number of particles of type k in generation t with i offspring of type 1 and j offspring of type 2.

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$$\tilde{\tilde{\Im}}(n) = \left\{\xi_j^k(t, l); k, j \in T; t = 1, \cdots, n-1; l = 1, \cdots, Z_j(t)\right\}$$

is the information, based on the observation of the entire family tree.

MPSD

$$p_{ij}^{k} = \frac{a_k(i,j)\theta_{1k}^{i}\theta_{2k}^{j}}{A_k(\theta_{1k}^{i}, \theta_{2k}^{i})}; (i,j) \in \Im_k$$

$$A_k(\theta_{1k}^{i}, \theta_{2k}^{i}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_k(i,j)\theta_{1k}^{i}\theta_{2k}^{j}$$

Doble Poisson

$$A_k(\theta_{1k}^i, \theta_{2k}^i) = \exp(\theta_{1k} + \theta_{2k})$$

$$p_{ij}^k = \frac{\theta_{1k}^i e^{-\theta_{1k}} \theta_{2k}^j e^{-\theta_{2k}}}{i!j!}$$

Family tree reconstruction

Gonzáles et al. [2]

$$P(\tilde{\tilde{\Im}}(n) \mid \Im(n), \theta_{11}, \theta_{21}) = \prod_{t=0}^{n-1} P(\tilde{\tilde{\Im}}(n) \mid Z(t), Z(t-1), \theta_{11}, \theta_{21})$$

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$$P(\tilde{\Im}(n) \mid Z(t), Z(t-1), \theta_{11}, \theta_{21}) = \frac{\prod\limits_{k=1,2} \prod\limits_{l=1}^{Z_k(t)} p^k_{\xi_1^k(t,l), \xi_2^k(t,l)}}{P(Z(t+1) \mid Z(t))}$$

$$P(\tilde{\Im}(n) \mid Z(t), Z(t-1), \theta_{11}, \theta_{21}) =$$

$$MN\left(\sum_{s=1}^{Z_1(s)} \xi_1^1(t,s), \sum_{s=1}^{Z_2(s)} \xi_2^1(t,s), \frac{Z_1(t)\theta_{11}}{Z_1(t)\theta_{11} + Z_2(t)\theta_{21}}, \frac{Z_2(t)\theta_{21}}{Z_1(t)\theta_{11} + Z_2(t)\theta_{21}}\right) \times$$

$$MN\left(\sum_{s=1}^{Z_1(s)} \xi_1^2(t,s), \sum_{s=1}^{Z_2(s)} \xi_2^2(t,s), \frac{Z_1(t)\theta_{12}}{Z_1(t)\theta_{12} + Z_2(t)\theta_{22}}, \frac{Z_2(t)\theta_{22}}{Z_1(t)\theta_{12} + Z_2(t)\theta_{22}}\right) \times$$

$$MN\left(\sum_{s=1}^{Z_1(s)} \xi_1^1(t,s), \frac{1}{Z_1(t)}\right) \times MN\left(\sum_{s=1}^{Z_2(s)} \xi_2^1(t,s), \frac{1}{Z_2(t)}\right) \times$$

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Reconstruction algorithm

1. Type 1 particles

$$Z_1(t+1) = \sum_{s=1}^{Z_1(s)} \xi_1^1(t,s) + \sum_{s=1}^{Z_2(s)} \xi_2^1(t,s),$$

   according MN (binomial) (1-st row)

2. Type 2 particles

$$Z_2(t+1) = \sum_{s=1}^{Z_1(s)} \xi_1^2(t,s) + \sum_{s=1}^{Z_2(s)} \xi_2^2(t,s),$$

   according MN (binomial) (2-nd row)

Reconstruction algorithm

3. Using $\sum_{s=1}^{Z_1(s)} \xi_1^1(t,s)$, generate $\xi_1^1(t,s)$ using $MN\left(\sum_{s=1}^{Z_1(s)} \xi_1^1(t,s), \frac{1}{Z_1(t)}\right)$
4. By analogy for $\xi_2^1(t,s)$
5. By analogy for $\xi_1^2(t,s)$
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Estimation of the parameters. ML

$$L(\tilde{\tilde{\Im}}(n) \mid \theta_{ik}; i, k = 1, 2) = \prod_{k=1}^{2} \prod_{t=1}^{n-1} \prod_{(i,j) \in \Im_k} p_{ij}^{k} =$$

$$= \prod_{k=1}^{2} \prod_{t=1}^{n-1} \prod_{(i,j) \in [\Im(k)]} \left[\frac{a_k(i,j)}{A_k(\theta_{1k}, \theta_{2k})} \theta_{1k}^{i} \theta_{2k}^{j} \right]^{Z_k(t,(i,j))}$$

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$$E\xi_k^1(t,l) = \frac{\theta_{1k}}{A_k(\theta_{1k},\theta_{2k})}\frac{\partial A_k(\theta_{1k},\theta_{2k})}{\partial\theta_{1k}}.$$

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$$E\xi_k^i(t,l) = \hat{m}_{ik} = \frac{\sum_{s=1}^{n} Z_i^k(s)}{\sum_{s=1}^{n-1} Z_k(s)}$$

Robust estimation. WLTE

Vandev & Nejkov [7].

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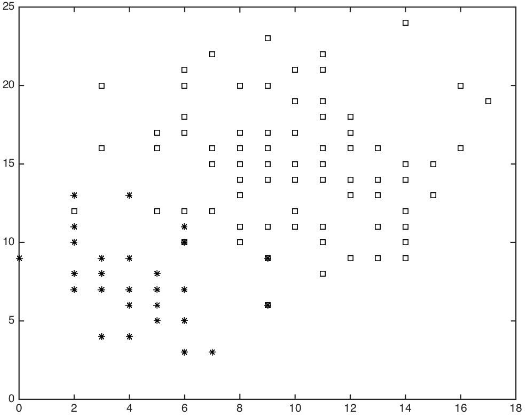
$$f_{\nu(1)}(\theta) \leq \ f_{\nu(2)}(\theta) \leq \cdots \leq f_{\nu(n)}(\theta)$$

$$f_{\nu(h)}(\theta) = -\log \phi(x_h, \theta)$$

for p.d.f $\phi(x, \theta)$ and a permutation $\nu = (\nu(1), \cdots, \nu(n))$.

An example



	θ_1	θ_2
True	10	15
Total	8.5769	13.5846
Robust	9.8800	15.3600
Outliers	4.2333	7.6667

Breakdown point

$BPoint(E) = \frac{1}{n}\max\{m : sup||E(X_m)|| < \infty\}$, where X_m is a sample, obtained from the sample X by replacing any m of the observations by arbitrary values, Hampel et al. (1986).

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$g(\theta)$ is subcompact if $L_g(C) = \{\theta : g(\theta) \leq C\}$ are compacts for any C.

Robustness, BPSD

$$N_h = (X_h, Y_h);\ h = 1, \cdots, n;\ w_h = 1$$

$$(\theta_1 \hat{} \theta_2) = \underset{(\theta_1,\theta_2)\in\Theta}{argmin} \sum_{h=1}^{k} - \log \left\{ \frac{A(\theta_1, \theta_2)}{a(X_{\nu(h)}, Y_{\nu(h)})\theta_1^{X_{\nu(h)}} \theta_2^{Y_{\nu(h)}}} \right\}$$

$$-log \frac{A(\theta_1, \theta_2)}{a(X_{\nu(1)}, Y_{\nu(1)})\theta_1^{X_{\nu(1)}} \theta_2^{Y_{\nu(1)}}} \leq \cdots \leq -log \frac{A(\theta_1, \theta_2)}{a(X_{\nu(n)}, Y_{\nu(n)})\theta_1^{X_{\nu(n)}} \theta_2^{Y_{\nu(n)}}}$$

- $\Im$ is the set of all posible values of the random vector (X, Y);
- $\Im^1$ the set of all possible values of the random variable X, while $\Im^2$ is the set of all possible values of the random variable Y;
- Υ is the convergence region of the series $A(\theta_1, \theta_1)$, and $\partial\Upsilon$ is its boundary;
- $MinVX \in \Im^1$ is the minimal value of the random variable X, and $MinVY \in \Im^2$ - the minimal value of the random variable Y;
- $MaxVX \in \Im^1$ is the maximal value of the random variable X (which may be finite or infinity), and $MaxVY \in \Im^2$ - the maximal value of the random variable Y;
- If $(MinVX, MinVY) \in \Im$, then $N_{MinVX,MinVY}$ is the observed number vectors in the sample with minimal values of the coordinates;
- If $MaxVX < \infty$, $MaxVY < \infty$ and $(MaxVX, MaxVY) \in \Im$, then $N_{MaxVX,MaxVY}$ is the observed number vectors in the sample with maximal values of the coordinates;

Lemma

[1]
*Let us consider a sample of n independent bivariate power series distributed observations. For the breakdown point of the $LTE(k)$ estimator (**??**) the following statements hold:*

1. *If* $|\Im| = \infty$, $(\theta_1, \theta_2) \in \Upsilon$, $\lim_{\theta_1,\theta_2 \to \partial\Upsilon} \frac{A(\theta_1,\theta_2)}{\theta_1^i \theta_2^j} = \infty \; \forall (i,j) \in \Im$ *except for* $(i,j) = (MinVX, MinVY)$, *then the* $WLT(K)$ *estimator exists and its breakdownpoint is not less than* $[n-k]/n$ *if* $n \geq 3(N_{MinVX,MinVY} + 1)$, $[n + N_{MinVX,MinVY} + 1]/2 \leq k \leq n - N_{MinVX,MinVY} - 1$.
2. *If* $|\Im| < \infty$; $\theta_1 \in (0,\infty)$, $\theta_2 \in (0,\infty)$, *then the* $WLT(K)$ *estimator exists and its breakdown point is not less than* $[n-k]/n$, *if* $n \geq 3(max\{N_{MinVX,MinVY}, N_{MaxVX,MaxVY}\} + 1)$, *and* $[n + max\{N_{MinVX,MinVY}, N_{MaxVX,MaxVY}\} + 1]/2 \leq K \leq n - max\{N_{MinVX,MinVY}, N_{MaxVX,MaxVY}\} - 1$.

   If $(MaxVX, MaxVY) \notin \Im$, *then the statement in 1. hold.*
3. *If* $(\theta_1, \theta_2) \in \Phi \subset \Upsilon_1$, *where* Φ *is a compact set, then the* $WLT(k)$ *estimator exists and its breakdown point is not less than* $[n-k]/n$ *if* $n \geq 3$, $[n+1]/2 \leq k \leq n-1$.

LTE estimation

1. Estimating the vector θ, for example using the Harris estimators on an available subset of the family tree, free of outliers.
2. Generating a family tree, using the observed generation sizes and the parameter value from 1.
3. Estimating the vector θ, using the $WLT(K)$ estimator over the entire family tree, and detecting the 'errors'.

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This algorithm can be bootstrapped large number of times.

EM estimation. M-step

[4]

$$\log L(\tilde{\tilde{\Im}}(n) \mid \theta_{ik}; i, k = 1, 2) = \sum_{k=1}^{2} \sum_{t=1}^{n-1} \sum_{(i,j) \in [\Im_k]} Z_k(t, (i, j)) \times$$

$$\times \left(\log \theta_{1k}^{i} + \log \theta_{2k}^{j} - \theta_{1k} - \theta_{2k} - \log i! j! \right)$$

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$$\times \left(\log \theta_{1k}^{i} + \log \theta_{2k}^{j} - \theta_{1k} - \theta_{2k} - \log i!j! \right)$$

$$E_Z \log L(\tilde{\tilde{\Im}}(n) \mid \theta_{ik}; i,k=1,2) \approx$$

$$\approx \sum_{k=1}^{2} \sum_{(i,j)\in[\Im_k]} EZ_k(i,j) \times (i \log \theta_{1k} + j \log \theta_{2k} - \theta_{1k} - \theta_{2k})$$

.

EM estimation. E-step

$$EZ_k(i,j) = \sum_{t=1}^{n-1} Z_1(t)Z_2(t)\hat{p}_{ij}^k$$

where

$$\hat{p}_{ij}^k = \frac{\hat{\theta}_{1k}^i e^{-\hat{\theta}_{1k}} \hat{\theta}_{2k}^j e^{-\hat{\theta}_{2k}}}{i!j!}.$$

EM algorithm

1. Set a starting values for $\{\theta_{ik}, i, k = 1, 2\}$
2. Perform E-step - calculate $EZ_k(i, j)$
3. Substitute $EZ_k(i, j)$ in $E_Z \log L(\tilde{\tilde{\Im}}(n) \mid \theta_{ik}; i, k = 1, 2)$
4. Perform M-step - maximize $E_Z \log L(\tilde{\tilde{\Im}}(n) \mid \theta_{ik}; i, k = 1, 2)$ and obtain new $\{\theta_{ik}, i, k = 1, 2\}$
5. Go to 2. if the $E_Z \log L$ is changed.

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