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ON SUBCRITICAL BRANCHING PROCESSES IN RANDOM ENVIRONMENT

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Let $\{\xi_n, n \in \mathbb{N}_0\}$ be a *branching process in a random environment* (BPRE). It is defined by a sequence of independent and identically distributed generating functions $\{f_n(s), n \in \mathbb{N}\}$.

Note that ξ_n is the size of the nth generation (we assume that $\xi_0 = 1$). The generating function $f_n(s)$, $s \in [0, 1]$, defines the reproduction law for the particles in the $(n-1)$th generation, $n \in \mathbb{N}$.

Assuming that $f_1'(1) \in (0,+\infty)$ a.s., we set $X_i = \ln f_i'(1)$ for $i \in \mathbb{N}$. Note that $X_1, X_2, \ldots$ are independent and identically distributed random variables. Introduce the *associated random walk*

$$S_0 = 0,\ S_n = \sum_{i=1}^{n} X_i,\ n \in \mathbb{N}.$$

Suppose that the process $\{\xi_n\}$ is *subcritical*, i.e. $\mathbf{E}X_1 < 0$, and there exists a positive number $\varkappa \in [0,1]$ such that

$$\mathbf{E}e^{\varkappa X_1} = 1, \ \mathbf{E}\left|X_1\right| e^{\varkappa X_1} < +\infty. \tag{1}$$

Condition (1) is classical for random walk with a negative drift and allows one to pass to *conjugate random walk* with a positive drift.

In addition, we assume that

$$\mathbf{E}\xi_1 \ln(\xi_1 + 1)\, e^{(\varkappa-1)X_1} < +\infty, \tag{2}$$

and if $\varkappa \geq 1$, then there exists a number $p > \varkappa$ such that

$$\mathbf{E}\left(\xi_1^p e^{(\varkappa-p)X_1}\right) < +\infty. \tag{3}$$

Introduce the *first passage time* of the process $\{\xi_n\}$ to a level $x \geq 1$:

$$T_x = \min\{n : \xi_n > x\},$$

and the *lifetime* of the process $\{\xi_n\}$:

$$T = \min\{n : \xi_n = 0\}.$$

It is known (see [1]) that if conditions (1)-(3) hold, then, as $x \to \infty$,

$$\mathbf{P}\left(T_x < \infty\right) \sim c_0 x^{-\varkappa},$$

where c_0 is a positive constant.

Set

$$a = \mathbf{E} X_1 e^{\varkappa X_1}, \ b = \mathbf{E} X_1.$$

There are *laws of large numbers* for the random values T_x and T (see [2]): if conditions (1)-(3) hold, then, as $x \to \infty$,

$$\left\{ \frac{T_x}{\ln x} \,\middle|\, T_x < \infty \right\} \xrightarrow{\mathbf{P}} \frac{1}{a}.$$

If, in addition, for some $\delta > 0$

$$\mathbf{E}\left(\ln^{+}\eta_1\right)^{1+\delta} < +\infty,$$

then, as $x \to \infty$,

$$\left\{\frac{T}{\ln x} \,\middle|\, T_x < \infty\right\} \xrightarrow{\mathbf{P}} \frac{1}{a} - \frac{1}{b}.$$

In addition, we assume that

$$\mathbf{E}\left(X_1^2 \exp\left(\varkappa X_1\right)\right) < +\infty. \quad (4)$$

Set

$$\sigma^2 = \mathbf{E}\left(X_1^2 \exp\left(\varkappa X_1\right)\right) - a^2.$$

Let $B = \{B(t),\ t \in [0,1]\}$ be a standard Brownian motion and the symbol $\overset{D}{\to}$ means convergence in distribution in the space $D[0,1]$ with the Skorokhod topology.

The following *functional limit theorem* for the first passage time to different levels is valid (see [3]).

Theorem 1. *If* $\{\xi_n,\ n \in \mathbb{N}_0\}$ *is a subcritical BPRE and conditions* (1)-(4) *hold, then, as* $x \to +\infty$,

$$\left\{ \frac{T_{x^t} - \frac{t \ln x}{a}}{\sigma \sqrt{\frac{\ln x}{a^3}}}, t \in [0,1] \,\middle|\, T_x < \infty \right\} \xrightarrow{D} B.$$

Also the following *functional limit theorem* for the size of different generations is valid.

Theorem 2. *If* $\{\xi_n,\ n \in \mathbb{N}_0\}$ *is a subcritical BPRE and conditions* (1)-(4) *hold, then, as* $y \to +\infty$,

$$\left\{ \frac{\ln \xi_{\left\lfloor \frac{ty}{a} \right\rfloor} - ty}{\sigma \sqrt{\frac{y}{a}}}, t \in [0,1) \middle| T_{e^y} < \infty \right\} \xrightarrow{D} B.$$

We notice that in the theorem 2 the variable t belongs to the set $[0, 1)$. The symbol $\xrightarrow{D}$ in this theorem means convergence in distribution in the space $D\,[0, u]$ with the Skorokhod topology for an arbitrary fixed $u \in (0, 1)$.

A few words about some ideas of the proof of the theorems. Denote Δ the set of probability measures on the set of nonnegative integers. Introduce on Δ the matric of total variation, then Δ is a complete separable metric space. Suppose that the probability measure Q_n corresponds to the generating function $f_n(s)$.

Go to the new probability measure $\widetilde{\mathbf{P}}$ and the corresponding mathematical expectation $\widetilde{\mathbf{E}}$ supposing for any $n \in \mathbb{N}$ and any measurable finite number function g defined on the set $\Delta^n \times \mathbb{N}_0^n$ that

$$\widetilde{\mathbf{E}} g\left(Q_1, \ldots, Q_n; \xi_1, \ldots, \xi_n\right) =$$

$$= \mathbf{E} e^{\varkappa S_n} g\left(Q_1, \ldots, Q_n; \xi_1, \ldots, \xi_n\right).$$

Concerning this measure the sequence $\{\xi_n\}$ is a supercritical BPRE and sequence $\{Q_n\}$ is the random environment of the BPRE. Notice that $\widetilde{\mathbf{E}} X_1 = a$.

The sequence $\{\xi_n / \exp S_n\}$ for a fixed random environment is a nonnegative martingale, therefore $\widetilde{\mathbf{P}}$-a.s. there is the limit

$$\lim_{n\to\infty} \frac{\xi_n}{\exp S_n} = W < +\infty.$$

For a supercritical BPRE the following two events are important:

$$D = \left\{ \lim_{n \to \infty} \xi_n = +\infty \right\},$$

$$D^* = \{W > 0\}.$$

The first event is called the *set of nonextinction* and the second event is called the *set of natural growth.* It is clear that

$$D^* \subset D.$$

If conditions (1)-(3) hold, then

$$\widetilde{\mathbf{P}}\,(D) > 0.$$

It is typical for supercritical branching processes.

The sets D and D^* are indistinguishable in probability sense, that is

$$\widetilde{\mathbf{P}}\,(D \triangle D^*) = 0.$$

The set D is approximated, on the one hand, by the set $G_k = \{\xi_k > 0\}$:

$$\lim_{k \to \infty} \widetilde{\mathbf{P}}\,(D \triangle G_k) = 0,$$

and, on the other hand, by the set $D_x = \{T_x < +\infty\}$:

$$\lim_{x \to +\infty} \widetilde{\mathbf{P}}\,(D \triangle D_x) = 0.$$

Set for $\varepsilon \in (0, 1/2)$

$$A_k(\varepsilon) = \left\{ \sup_{n:\, n>k} \left| \frac{e^{S_n}}{\xi_n} - \frac{e^{S_k}}{\xi_k} \right| \leq \varepsilon \frac{e^{S_k}}{\xi_k} \right\}.$$

It is clear that $A_k(\varepsilon) \cap D^*$ and D^* are close.

Introduce the random event

$$D_{x,k} = D \cap D_x \cap \{T_x > k\} \cap A_k(\varepsilon),$$

which has an important significance in the theory.

It turns out that D_x and G_k are approximated by the set $D_{x,k}$:

$$\lim_{k\to\infty} \limsup_{x\to+\infty} \widetilde{\mathbf{P}}\left(D_x \triangle D_{x,k}\right) = 0,$$

$$\lim_{k\to\infty} \limsup_{x\to+\infty} \widetilde{\mathbf{P}}\left(G_k \triangle D_{x,k}\right) = 0.$$

If the event $D_{x,k}$ is valid, then

$$\ln \xi_n \approx S_n,$$

and $\{S_n\}$ is a random walk with a positive drift, for which there is the well advanced *renewal theory*.

Recall some facts of this theory. Let $\{S_n\}$ be a random walk with positive drift a. Set for $x > 0$

$$\mathcal{T}_x = \min\{n : S_n > x\}.$$

The following law of large number is valid: almost sure, as $x \to \infty$,

$$\frac{\mathcal{T}_x}{x} \to \frac{1}{a}.$$

Suppose that the variance σ^2 of a step of the random walk is finite and positive. For arbitrary $x > 0$ introduce the random process Z_x:

$$Z_x(t) = \frac{\mathcal{T}_{tx} - \frac{tx}{a}}{\sigma\sqrt{\frac{x}{a^3}}}, \qquad t \geq 0.$$

The following functional limit theorem is valid (see [4])): as $x \to \infty$,

$$\{Z_x(t), \ t \in [0,1]\} \xrightarrow{D} B.$$

At last for the random walk $\{S_n\}$ the following functional limit theorem is valid: as $n \to \infty$,

$$\left\{ \frac{S_{\lfloor nt \rfloor} - nta}{\sigma\sqrt{n}},\ t \in [0,1] \right\} \xrightarrow{D} B.$$

The mentioned laws of large numbers and functional limit theorems for BPRE are corollaries of the corresponding theorems of the renewal theory.

References

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