Local limits of conditioned Galton-Watson trees

R. Abraham - J.F. Delmas

WBPA 2015

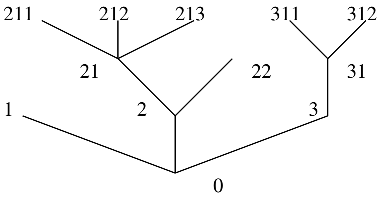
The set of discrete trees

Let $\mathcal{U} = \bigcup_{n=0}^{+\infty} (\mathbb{N}^*)^n$ be the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$.

A tree $\mathbf{t}$ is a sub-set of $\mathcal{U}$ such that

- $\emptyset \in \mathbf{t}$
- If $ui \in \mathbf{t}$, then $u \in \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists an integer $k_u(\mathbf{t})$ such that

$$ui \in \mathbf{t} \iff 0 \leq i \leq k_u(\mathbf{t}).$$



We denote by $\mathbb{T}$ the set of discrete trees.

The local topology

For $u = i_1 i_2 \ldots i_n \in \mathcal{U}$, we denote by $|u| = n$ the generation of the node u.
If $\mathbf{t} \in \mathbb{T}$, for every integer h, we define the truncation of the tree $\mathbf{t}$ at height h by

$$r_h(\mathbf{t}) = \{u \in \mathbf{t},\ |u| \leq h\}.$$

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A sequence of trees $(\mathbf{t}_n)$ converges locally to $\mathbf{t}$ if, for every height h,

$$r_h(\mathbf{t}_n) = r_h(\mathbf{t}) \quad \text{for } n \text{ large enough.}$$

A sequence of random trees (T_n) converges in law to a random tree T with respect to this distance if for every h and every tree $\mathbf{t} \in \mathbb{T}^{(h)}$,

$$\mathbb{P}(r_h(T_n) = \mathbf{t}) \longrightarrow \mathbb{P}(r_f(T) = \mathbf{t}).$$

Conditioning on non-extinction

Theorem (Kesten, 1986)

Let p be a critical or sub-critical offspring distribution.
Let τ_n be a random tree whose distribution is those of τ conditioned on $\{H(\tau) \geq n\}$.
Then

$$\tau_n \xrightarrow{(d)} \tau^*.$$

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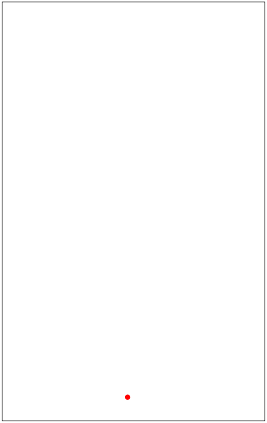

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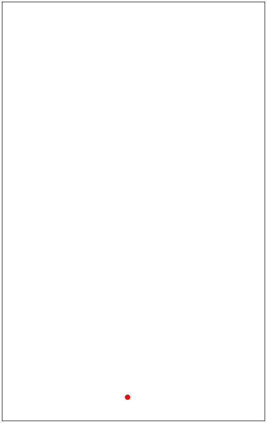

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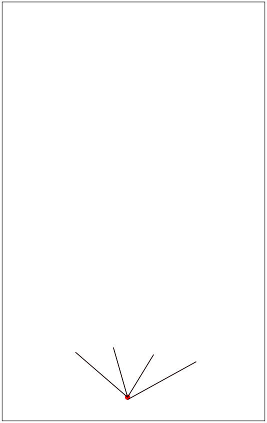

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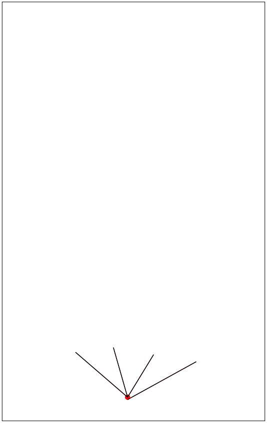

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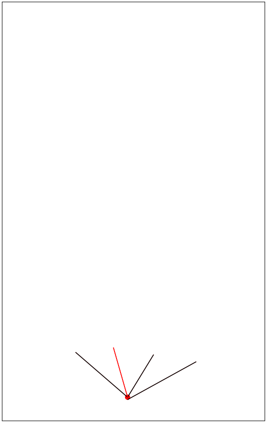

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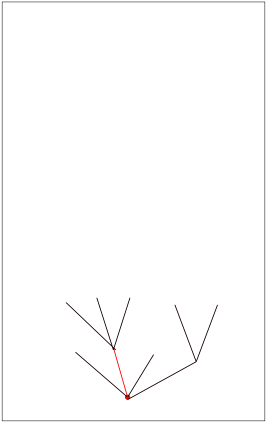

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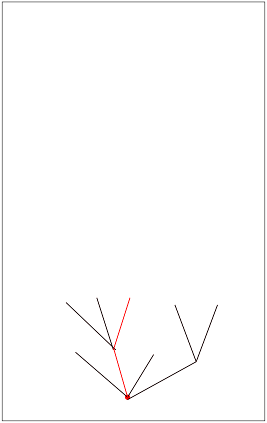

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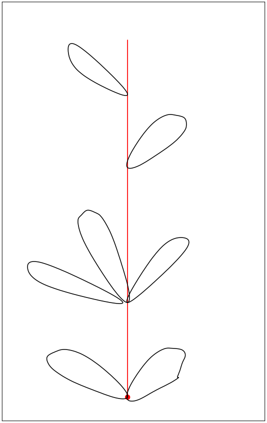

Other conditionings

We suppose that p is critical.

- Aldous-Pitman (1998) : conditioning on the total progeny of the tree.
- Curien-Kortchemski (2014) : conditioning on the number of leaves

In these cases, we still have

$$\tau_n \xrightarrow{(d)} \tau^*.$$

Another caracterization of local convergence

If $\mathbf{t}, \mathbf{t}' \in \mathbb{T}$ and $x \in \mathcal{L}_0(\mathbf{t})$, we denote by

$$\mathbf{t} \circledast_x \mathbf{t}' = \mathbf{t} \cup \{xv, v \in \mathbf{t}'\}$$

the grafting of $\mathbf{t}'$ on the leaf x of $\mathbf{t}$.

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Lemme

A sequence $(T_n, n \in \mathbb{N})$ of random trees that a.s. belong to $\mathbb{T}_0 \cup \mathbb{T}_1$ converges in distribution to a tree T that also a.s. belongs to $\mathbb{T}_0 \cup \mathbb{T}_1$ if and only if, for every finite tree $\mathbf{t}$ and every leaf x of $\mathbf{t}$,

$$\lim_{n \to +\infty} \mathbb{P}(T_n \in \mathbb{T}(\mathbf{t}, x)) = \mathbb{P}(T \in \mathbb{T}(\mathbf{t}, x)) \text{ et } \lim_{n \to +\infty} \mathbb{P}(T_n = \mathbf{t}) = \mathbb{P}(T = \mathbf{t}).$$

Functionals

Let $A : \mathbb{T} \longrightarrow \mathbb{N} \cup \{\infty\}$ be a real-valued function defined on $\mathbb{T}$ that is finite on $\mathbb{T}_0$.
We want to condition a critical or sub-critical Galton-Watson tree τ on $\{A(\tau) \geq n\}$ or on $\{A(\tau) = n\}$.
We will consider three different assumptions on the functional A:

- (Identity) $A(\mathbf{t} \circledast_x \tilde{\mathbf{t}}) = A(\tilde{\mathbf{t}})$

All these properties are supposed to hold for $A(\mathbf{t} \circledast_x \tilde{\mathbf{t}})$ large enough.

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- (Identity) $A(\mathbf{t} \circledast_x \tilde{\mathbf{t}}) = A(\tilde{\mathbf{t}})$
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- (Monotonicity) $A(\mathbf{t} \circledast_x \tilde{\mathbf{t}}) \geq A(\tilde{\mathbf{t}})$
- (Additivity) There exists a function $D(\mathbf{t}, x) \geq 0$ such that

$$A(\mathbf{t} \circledast_x \tilde{\mathbf{t}}) = A(\tilde{\mathbf{t}}) + D(\mathbf{t}, x).$$

All these properties are supposed to hold for $A(\mathbf{t} \circledast_x \tilde{\mathbf{t}})$ large enough.

Convergence with the identity assumption

Theorem

Let p be a **critical** offspring distribution, let τ be the associated Galton-Watson tree. We suppose that A satisfies the *Identity* assumption. Then

$$\mathrm{dist}(\tau | A(\tau) = n) \longrightarrow \mathrm{dist}(\tau^*)$$

and

$$\mathrm{dist}(\tau | A(\tau) \geq n) \longrightarrow \mathrm{dist}(\tau^*)$$

Conditioning on the maximum out-degree (X. He)

For a tree $\mathbf{t}$, we consider

$$A(\mathbf{t}) = \sup_{u \in \mathbf{t}} k_u(\mathbf{t})$$

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We deduce that, for every critical offspring distribution with unbounded support

$$\mathrm{dist}(\tau | A(\tau) = n) \longrightarrow \mathrm{dist}(\tau^*).$$

Convergence with the monotonicity assumption

Theorem (X. He, 2015)

Let p be a **critical** offspring distribution, let τ be the associated Galton-Watson tree. We suppose that A satisfies the *Monotonicity* assumption. Then

$$\mathrm{dist}(\tau | A(\tau) \geq n) \longrightarrow \mathrm{dist}(\tau^*)$$

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The height of the tree satisfies the monotonicity assumption, we hence recover Kesten's theorem.

Convergence with the additivity assumption

Theorem

Let p be a **critical** offspring distribution, let τ be the associated Galton-Watson tree. We suppose that A satisfies the *Additivity* assumption. We suppose moreover that

$$\limsup_{n\to+\infty} \frac{\mathbb{P}(A(\tau)=n+1)}{\mathbb{P}(A(\tau)=n)} \leq 1.$$

Then

$$\mathrm{dist}(\tau | A(\tau)=n) \longrightarrow \mathrm{dist}(\tau^*)$$

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Dwass formula(1969)

Let (W_k) be i.i.d. random variables distributed as $\#\tau$.
Let (X_k) be i.i.d. random variables with distribution p.
Then, for every $k > 0$ and every $n \geq k$,

$$\mathbb{P}(W_1 + \cdots + W_k = n) = \frac{k}{n}\mathbb{P}(X_1 + \cdots + X_n = n - k).$$

Conditioning on the total progeny, critical case (continuation)

Strong ratio theorem

Let Y be a $\mathbb{Z}$-valued random variable, aperiodic and centered.
Let S_n be the associated random walk:

$$S_n = \sum_{k=1}^{n} Y_k.$$

Then

$$\lim_{n\to+\infty} \frac{\mathbb{P}(S_n = \ell)}{\mathbb{P}(S_n = 0)} = \lim_{n\to+\infty} \frac{\mathbb{P}(S_{n+1} = 0)}{\mathbb{P}(S_n = 0)} = 1.$$

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Proposition

Let p be a critical aperiodic offspring distribution. Then

$$\mathrm{dist}(\tau \mid \#\tau = n) \longrightarrow \mathrm{dist}(\tau^*).$$

Nodes with a fixed number of offsprings

Let $\mathcal{A} \subset \mathbb{N}$.
For $\mathbf{t} \in \mathbb{T}$, we set

$$\mathcal{L}_{\mathcal{A}}(\mathbf{t}) = \{u \in \mathbf{t},\ k_u(\mathbf{t}) \in \mathcal{A}\}$$

and

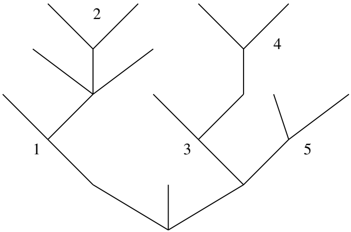
$$L_{\mathcal{A}}(\mathbf{t}) = \#\mathcal{L}_{\mathcal{A}}(\mathbf{t}).$$

Remark:

- $\mathcal{A} = \mathbb{N}$: total progeny
- $\mathcal{A} = \{0\}$: number of leaves
- $\mathcal{A} = \mathbb{N}^*$: number of internal nodes

Maping $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$ onto the nodes of a tree

$\mathcal{A} = \{2\}$.



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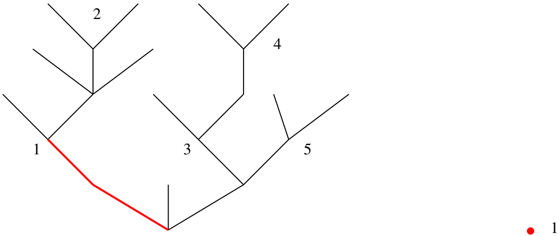

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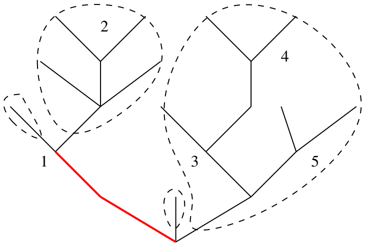

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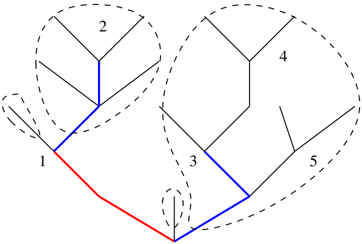

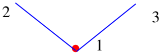

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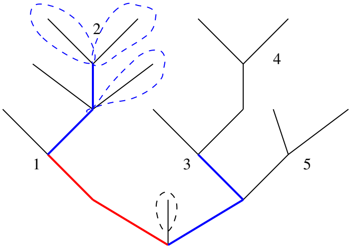

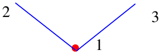

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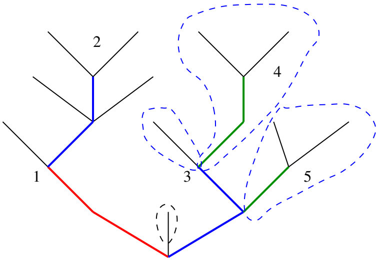

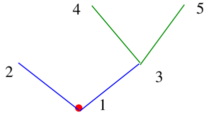

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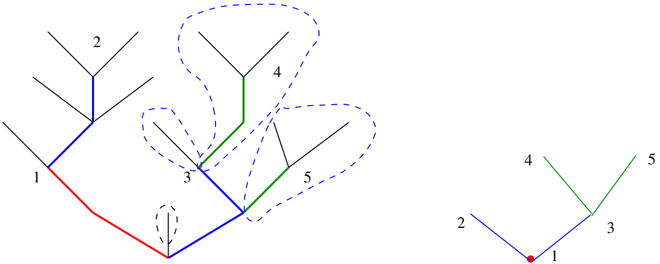

Lemme (Rizzolo 2014)

If τ is a critical Galton-Watson tree, then the tree $\tau_{\mathcal{A}}$ conditionally given $\{\mathcal{L}_{\mathcal{A}}(\tau) \neq \emptyset\}$ is also a critical Galton-Watson tree.

Conditioning on $L_{\mathcal{A}}(\tau)$

Proposition

Let p be a critical offspring distribution and $\mathcal{A} \subset \mathbb{N}$. Then

$$\mathrm{dist}(\tau | L_{\mathcal{A}}(\tau) = n) \longrightarrow \mathrm{dist}(\tau^*).$$

Sub-critical case : extension of the additive property

Theorem

Let p be a **sub-critical** offspring distribution with mean μ, let τ be the associated Galton-Watson tree. We suppose that A satisfies the *Additivity* assumption with $D(\mathbf{t}, x) = |x|$. We suppose moreover that

$$\limsup_{n\to+\infty} \frac{\mathbb{P}(A(\tau) = n+1)}{\mathbb{P}(A(\tau) = n)} \leq \mu.$$

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This theorem only applies for $A(\mathbf{t}) = H(\mathbf{t})$.

Sub-critical case: an equivalent offspring distribution for $L_{\mathcal{A}}$

We set

$$p_\theta(k) = \begin{cases} c_{\mathcal{A}}(\theta)\theta^k p(k) & \text{si } k \in \mathcal{A}, \\ \theta^{k-1} p(k) & \text{si } k \notin \mathcal{A} \end{cases}$$

with

$$c_{\mathcal{A}} = \frac{1 - \sum_{k \notin \mathcal{A}} \theta^{k-1} p(k)}{\sum_{k \in \mathcal{A}} \theta^k p(k)}.$$

We denote by I the set of θ for which p_θ is a probability distribution on $\mathbb{N}$.

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Proposition

For every $\theta \in I$ and every $n \in \mathbb{N}^*$,

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Remark:

- $\mathcal{A} = \mathbb{N}$: Kennedy 1975
- $\mathcal{A} = \{0\}$: Abraham-Delmas-He 2012

Conditioning on $L_{\mathcal{A}}$, generic case

Proposition

Let p be a sub-critical offspring distribution such that there exists some $\theta_c \in I$ such that p_{θ_c} is critical (such a distribution is said to be generic for $\mathcal{A}$). Then

- θ_c is unique.
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Conditioning on $L_{\mathcal{A}}$, generic case

Proposition

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If such a θ_c does not exist (non-generic case), then a condensation phenomenon appears, Jonnsson-Stefansson 2011, Janson 2012.

The condensation tree

If p is a sub-critical offspring distribution, we define the associated condensation tree $\tau^\infty(p)$ by

- The nodes have two types, normal and special.

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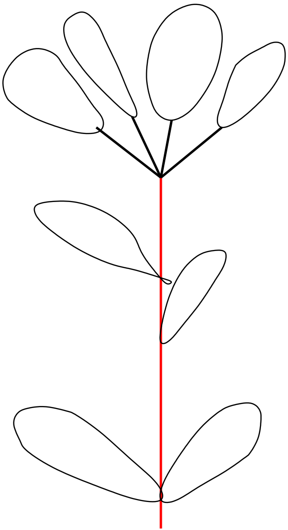
- When a special node has an infinite number of children, all of them are normal.

The condensation tree



Conditioning on $L_{\mathcal{A}}$, non-generic case

Theorem

Let p be a non-generic offspring distribution for $\mathcal{A}$. We set $\theta_\infty = \max I$. Then

$$\mathrm{dist}(\tau | L_{\mathcal{A}}(\tau) = n) \longrightarrow \mathrm{dist}(\tau^\infty(p_{\theta_\infty})).$$

Generic and non-generic distribution

Lemme

Let p be a sub-critical offspring distribution, let φ be its generating function, and let ρ be the radius of convergence of φ.

- If $\rho = +\infty$ or if $g'(\rho) \geq 1$, then p is generic for all $\mathcal{A}$.

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- If $1 < \rho < +\infty$ and $g'(\rho) < 1$ then p is non-generic for $\mathcal{A}$ iff

$$\mathbb{E}[Y | Y \in \mathcal{A}] < \frac{1 - \varphi'(\rho)}{\rho - \varphi(\rho)}$$

where Y is distributed according to $p_{\mathbb{N},\rho}$.
In, p is non-generic for $\{0\}$ but is generic for $\{k\}$ for k large enough.

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  In, p is non-generic for $\{0\}$ but is generic for $\{k\}$ for k large enough.
- There exists some distributions that are generic for $\mathbb{N}$ and not for $\{0\}$.