Limit theorem for critical inhomogeneous branching processes with immigration

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Abstract

A zero start inhomogeneous branching process with immigration (IBPI) $(X_n)_{n\in\mathbb{Z}_+}$ is defined as

$$X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j} + \varepsilon_n, \quad n \in \mathbb{N}, \qquad X_0 = 0,$$

where $\{\xi_{n,j}, \varepsilon_n : n, j \in \mathbb{N}\}$ are independent non-negative integer-valued random variables such that $\{\xi_{n,j} : j \in \mathbb{N}\}$ are identically distributed for each $n \in \mathbb{N}$. Assume that $m_n := \mathsf{E}\xi_{n,1}$, $\lambda_n := \mathsf{E}\varepsilon_n$, $\sigma_n^2 := \mathsf{Var}\xi_{n,1}$, $b_n^2 := \mathsf{Var}\varepsilon_n$ are finite for all $n \in \mathbb{N}$. The process $(X_n)_{n\in\mathbb{Z}_+}$ is called (nearly) critical if $m_n \to 1$ as $n \to \infty$. Introduce the random step functions

$$\mathcal{X}^{(n)}(t) := X_{\lfloor nt \rfloor} \qquad \text{for } t \in \mathbb{R}_+, \ n \in \mathbb{N}.$$

We prove the following generalization of a result of Wei and Winnicki [1].

Theorem. Suppose that $\sum_{n=1}^{\infty} |m_n - 1| < \infty$; $\sigma_n^2 \to \sigma^2 \geq 0$, $\lambda_n \to \lambda \geq 0$, $b_n^2 \to b^2 \geq 0$, and $n^{-2}\sum_{k,j=1}^{n} \mathsf{E}\left(|\xi_{k,j} - m_k|^2 \mathbb{1}_{\{|\xi_{k,j}-m_k|>\theta n\}}\right) \to 0$ for all $\theta > 0$ as $n \to \infty$. Then

$$n^{-1}\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{X} \qquad \text{as } n \to \infty,$$

that is, weakly in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, where $\big(\mathcal{X}(t)\big)_{t\in\mathbb{R}_+}$ is the unique solution of a stochastic differential equation (SDE)

$$\mathrm{d}\mathcal{X}(t) = \lambda\,\mathrm{d}t + \sigma\sqrt{\mathcal{X}_+(t)}\,\mathrm{d}W(t), \qquad t \in \mathbb{R}_+,$$

with initial condition $\mathcal{X}(0) = 0$, where $x_+ := \max\{x, 0\}$ and $(W(t))_{t\in\mathbb{R}_+}$ is a standard Wiener process.

References

[1] Wei, C.Z. and Winnicki, J. (1990). *Estimation of the means in the branching process with immigration.* Ann. Statist., 18, 1757–1773.